

ON THE FORMAL GROUP LAWS OF UNORIENTED AND COMPLEX COBORDISM THEORY

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In this note we outline a connection between the generalized cohomology theories of unoriented cobordism and (weakly-) complex cobordism and the theory of formal commutative groups of one variable [4], [5]. This connection allows us to apply Cartier's theory of typical group laws to obtain an explicit decomposition of complex cobordism theory localized at a prime p into a sum of Brown-Peterson cohomology theories [1] and to determine the algebra of cohomology operations in the latter theory.

1. Formal group laws. If R is a commutative ring with unit, then by a *formal* (commutative) *group law* over R one means a power series $F(X, Y)$ with coefficients in R such that

- (i) $F(X, 0) = F(0, X) = X$,
- (ii) $F(F(X, Y), Z) = F(X, F(Y, Z))$,
- (iii) $F(X, Y) = F(Y, X)$. We let $I(X)$ be the "inverse" series satisfying $F(X, I(X)) = 0$ and let

$$\omega(X) = dX/F_2(X, 0)$$

be the normalized invariant differential form, where the subscript 2 denotes differentiation with respect to the second variable. Over $R \otimes \mathcal{Q}$, there is a unique power series $l(X)$ with leading term X such that

$$(1) \quad l(F(X, Y)) = l(X) + l(Y).$$

The series $l(X)$ is called the *logarithm* of F and is determined by the equations

$$(2) \quad \begin{aligned} l'(X)dX &= \omega(X), \\ l(0) &= 0. \end{aligned}$$

2. The formal group law of complex cobordism theory. By *complex cobordism theory* $\Omega^*(X)$ we mean the generalized cohomology theory associated to the spectrum MU . If E is a complex vector bundle of dimension n over a space X , we let $c_i^{\mathcal{Q}}(E) \in \Omega^{2i}(X)$, $1 \leq i \leq n$ be

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the Chern classes of E in the sense of Conner-Floyd [3]. Since $\Omega^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = \Omega^*(pt) [[x, y]]$, where $x = c_1^\Omega(\mathcal{O}(1)) \otimes 1$, $y = 1 \otimes c_1^\Omega(\mathcal{O}(1))$ and $\mathcal{O}(1)$ is the canonical line bundle on $\mathbb{C}P^\infty$, there is a unique power series $F^\Omega(X, Y) = \sum a_{kl} X^k Y^l$ with $a_{kl} \in \Omega^{2-2k-2l}(pt)$ such that

$$(3) \quad c_1^\Omega(L_1 \otimes L_2) = F^\Omega(c_1^\Omega(L_1), c_1^\Omega(L_2))$$

for any two complex line bundles with the same base. The power series F^Ω is a formal group law over $\Omega^{ev}(pt)$.

THEOREM 1. *Let E be a complex vector bundle of dimension n , let $f: PE' \rightarrow X$ be the associated projective bundle of lines in the dual E' of E , and let $\mathcal{O}(1)$ be the canonical quotient line bundle on PE' . Then the Gysin homomorphism $f_*: \Omega^q(PE') \rightarrow \Omega^{q-2n+2}(X)$ is given by the formula*

$$(4) \quad f_*(u(\xi)) = \text{res} \frac{u(Z)\omega(Z)}{\prod_{j=1}^n F^\Omega(Z, I\lambda_j)} .$$

Here $u(Z) \in \Omega(X)[Z]$, $\xi = c_1^\Omega(\mathcal{O}(1))$, ω and I are the invariant differential form and inverse respectively for the group law F^Ω , and the λ_j are the dummy variables of which $c_q^\Omega(E)$ is the q th-elementary symmetric function.

The hardest part of this theorem is to define the residue; we specialize to dimension one an unpublished definition of Cartier, which has also been used in a related form by Tate [7].

Applying the theorem to the map $f: \mathbb{C}P^n \rightarrow pt$, we find that the coefficient of $X^n dX$ in $\omega(X)$ is P_n , the cobordism class of $\mathbb{C}P^n$ in $\Omega^{-2n}(pt)$. From (2) we obtain the

COROLLARY (MYSHENKO [6]). *The logarithm of the formal group law of complex cobordism theory is*

$$(5) \quad l(X) = \sum_{n \geq 0} P_n \frac{X^{n+1}}{n+1} .$$

3. The universal nature of cobordism group laws.

THEOREM 2. *The group law F^Ω over $\Omega^{ev}(pt)$ is a universal formal (commutative) group law in the sense that given any such law F over a commutative ring R there is a unique homomorphism $\Omega^{ev}(pt) \rightarrow R$ carrying F^Ω to F .*

PROOF. Let F_u over L be a universal formal group law [5] and let $h: L \rightarrow \Omega^{ev}(pt)$ be the unique ring homomorphism sending F_u to F^Ω . The law F_u over $L \otimes \mathcal{Q}$ is universal for laws over \mathcal{Q} -algebras. Such a

law is determined by its logarithm series which can be any series with leading term X . Thus if $\sum p_n X^{n+1}/n+1$ is the logarithm of F_u , $L \otimes \mathcal{Q}$ is a polynomial ring over \mathcal{Q} with generators p_i . By (5) $hp_i = P_i$, so as $\Omega^*(pt) \otimes \mathcal{Q} \cong \mathcal{Q}[P_1, P_2, \dots]$, it follows that $h \otimes \mathcal{Q}$ is an isomorphism.

By Lazard [5, Theorem II], L is a polynomial ring over \mathcal{Z} with infinitely many generators; in particular L is torsion-free and hence h is injective. To prove surjectivity we show $h(L)$ contains generators for $\Omega^*(pt)$. First of all $hp_n = P_n \in h(L)$ because $p_n \in L$ as it is the n th coefficient of the invariant differential of F_u . Secondly we must consider elements of the form $[M_n]$ where M_n is a nonsingular hypersurface of degree k_1, \dots, k_r in $CP^{n_1} \times \dots \times CP^{n_r}$. Let π be the map of this multiprojective space to a point. Then $[M_n] = \pi_* c_1^{\Omega}(L_1^{k_1} \otimes \dots \otimes L_r^{k_r})$, where L_j is the pull-back of the canonical line bundle on the j th factor. The Chern class of this tensor product may be written using the formal group law F^{Ω} in the form $\sum \pi^* a_{i_1 \dots i_r} z_1^{i_1} \dots z_r^{i_r}$, where $0 \leq i_j \leq n_j$, $1 \leq j \leq r$, where $z_i = c_1^{\Omega}(L_i)$, and where $a_{i_1 \dots i_r} \in h(L)$. Since

$$\pi_* z_1^{i_1} \dots z_r^{i_r} = \prod_{j=1}^r P_{n_j - i_j}$$

also belongs to $h(L)$, it follows that $[M_n] \in h(L)$. Thus h is an isomorphism and the theorem is proved.

We can also give a description of the unoriented cobordism ring using formal group laws. Let $\eta^*(X)$ be the unoriented cobordism ring of a space X , that is, its generalized cohomology with values in the spectrum MO . There is a theory of Chern (usually called Whitney) classes for real vector bundles with $c_i(E) \in \eta^i(X)$. The first Chern class of a tensor product of line bundles gives rise to a formal group law F^{η} over the commutative ring $\eta^*(pt)$. Since the square of a real line bundle is trivial, we have the identity

$$(6) \quad F^{\eta}(X, X) = 0.$$

THEOREM 3. *The group law F^{η} over $\eta^*(pt)$ is a universal formal (commutative) group law over a ring of characteristic two satisfying (6).*

4. Typical group laws (after Cartier [2]). Let F be a formal group law over R . Call a power series $f(X)$ with coefficients in R and without constant term a *curve* in the formal group defined by the law. The set of curves forms an abelian group with addition $(f +^F g)(X) = F(f(X), g(X))$ and with operators

$$\begin{aligned}
 ([r]f)(X) &= f(rX) & r \in R \\
 (V_n f)(X) &= f(X^n) & n \geq 1 \\
 (F_n f)(X) &= \sum_{i=1}^n f(\zeta_i X^{1/n}) & n \geq 1,
 \end{aligned}$$

where the ζ_i are the n th roots of 1. The set of curves is filtered by the order of a power series and is separated and complete for the filtration.

If R is an algebra over $\mathbf{Z}_{(p)}$, the integers localized at the prime p , then a curve is said to be *typical* if $F_q f = 0$ for any prime $q \neq p$. If R is torsion-free then it is the same to require that the series $l(f(X))$ over $R \otimes \mathbf{Q}$ has only terms of degree a power of p , where l is the logarithm of F . The group law F is said to be a *typical law* if the curve $\gamma_0(X) = X$ is typical. There is a canonical change of coordinates rendering a given law typical. Indeed let c_F be the curve

$$(7) \quad c_F^{-1} = \sum_{(n,p)=1} \frac{\mu(n)}{n} V_n F_n \gamma_0$$

where the sum as well as division by n prime to p is taken in the filtered group of curves and where μ is the Möbius function. Then the group law $(c_F * F)(X, Y) = c_F(F(c_F^{-1}X, c_F^{-1}Y))$ is typical.

5. Decomposition of $\Omega_{(p)}^*$. For the rest of this paper p is a fixed prime. Let $\Omega_{(p)}^*(X) = \Omega^*(X) \otimes \mathbf{Z}_{(p)}$ and let $\xi = c_F \Omega$. Then $\xi(Z)$ is a power series with leading term Z with coefficients in $\Omega_{(p)}^*(pt)$, so there is a unique natural transformation $\hat{\xi}: \Omega_{(p)}^*(X) \rightarrow \Omega_{(p)}^*(X)$ which is stable, a ring homomorphism, and such that

$$\hat{\xi} c_1^\Omega(L) = \xi(c_1^\Omega(L))$$

for all line bundles L .

THEOREM 4. *The operation $\hat{\xi}$ is homogeneous, idempotent, and its values on $\Omega_{(p)}^*(pt)$ are*

$$\begin{aligned}
 \hat{\xi}(P_n) &= P_n \quad \text{if } n = p^a - 1 \text{ for some } a \geq 0, \\
 &= 0 \quad \text{otherwise.}
 \end{aligned}$$

Let $\Omega T^(X)$ be the image of $\hat{\xi}$. Then there are canonical ring isomorphisms*

$$(8) \quad \Omega T^*(pt) \otimes_{\Omega_{(p)}^*(pt)} \Omega_{(p)}^*(X) \cong \Omega T^*(X),$$

$$(9) \quad \Omega_{(p)}^*(pt) \otimes_{\Omega T^*(pt)} \Omega T^*(X) \cong \Omega_{(p)}^*(X).$$

ΩT^* is the generalized cohomology theory associated to the Brown-Peterson spectrum [1] localized at p .

It is also possible to apply typical curves to unoriented cobordism theory where the prime involved is $p=2$. One defines similarly an idempotent operator ξ whose image now is $H^*(X, \mathbf{Z}/2\mathbf{Z})$; there is also a canonical ring isomorphism

$$\eta^*(pt) \otimes H^*(X, \mathbf{Z}/2\mathbf{Z}) \simeq \eta^*(X)$$

analogous to (9).

6. **Operations in ΩT^* .** If $\pi: \Omega_{(p)}^* \rightarrow \Omega T^*$ is the surjection induced by ξ , then π carries the Thom class in $\Omega_{(p)}^*(MU)$ into one for ΩT^* . As a consequence ΩT^* has the usual machinery of characteristic classes with $c_i^{\Omega T^*}(E) = \pi c_i^{\Omega}(E)$ and $F^{\Omega T^*} = \pi F^{\Omega}$. Let $t = (t_1, t_2, \dots)$ be an infinite sequence of indeterminates and set

$$\phi_t(X) = \sum_{n \geq 0}^{F^{\Omega T^*}} t_n X^{p^n} \quad t_0 = 1$$

where the superscript on the summation indicates that the sum is taken as curves in the formal group defined by $F^{\Omega T^*}$. There is a unique stable multiplicative operation $(\phi_t^{-1})^\wedge: \Omega^*(X) \rightarrow \Omega T^*(X)[t_1, t_2, \dots]$ such that

$$(\phi_t^{-1})^\wedge c_1^\Omega(L) = \phi_t^{-1}(c_1^{\Omega T^*}(L))$$

for all line bundles L . This operation can be shown using (8) to kill the kernel of π and hence it induces a stable multiplicative operation

$$r_t: \Omega T^*(X) \rightarrow \Omega T^*(X)[t_1, t_2, \dots].$$

Writing

$$r_t(x) = \sum_{\alpha} r_{\alpha}(x) t^{\alpha} \quad \text{if } x \in \Omega T^*(X)$$

where the sum is taken over all sequences $\alpha = (\alpha_1, \alpha_2, \dots)$ of natural numbers all but a finite number of which are zero, we obtain stable operations

$$r_{\alpha}: \Omega T^*(X) \rightarrow \Omega T^*(X).$$

THEOREM 5. (i) r_{α} is a stable operation of degree $2 \sum_i \alpha_i (p^i - 1)$. Every stable operation may be uniquely written as an infinite sum

$$\sum_{\alpha} u_{\alpha} r_{\alpha} \quad u_{\alpha} \in \Omega T^*(pt)$$

and every such sum defines a stable operation.

(ii) If $x, y \in \Omega T^*(X)$, then

$$r_\alpha(xy) = \sum_{\beta+\gamma=\alpha} r_\beta(x)r_\gamma(y).$$

(iii) The action of r_α on $\Omega T^*(pt)$ is given by

$$r_t(P_{p^{n-1}}) = \sum_{h=0}^n p^{n-h} P_{p^{h-1}} t_n^{p^h}.$$

(iv) If $t' = (t'_1, t'_2, \dots)$ is another sequence of indeterminates, then the compositions $r_\alpha \circ r_\beta$ are found by comparing the coefficients of $t^\alpha t'^\beta$ in

$$r_t \circ r_{t'} = \sum_\gamma \Phi(t, t')^\gamma r_\gamma$$

where $\Phi = (\Phi_1(t_1; t'_1), \Phi_2(t_1, t_2; t'_1, t'_2), \dots)$ is the sequence of polynomials with coefficients in $\Omega T^*(pt)$ in the variables t_i and t'_i obtained by solving the equations

$$\sum_{h=0}^N p^{N-h} P_{p^{h-1}} \Phi_{N-h}^{p^h} = \sum_{k+m+n=N} p^{m+n} P_{p^{k-1}} t_m^{p^k} t_n^{p^{k+m}}.$$

This theorem gives a complete description of the algebra of operations in ΩT^* . The situation is similar to that for Ω^* except the set of $\mathbf{Z}_{(p)}$ -linear combinations of the r_α 's is not closed under composition.

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