THE ASYMPTOTIC MANIFOLD OF A NONLINEAR SYSTEM OF DIFFERENTIAL EQUATIONS

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Communicated by Fred Brauer, June 23, 1969

The linear system of differential equations

$$dy/dt = A(t)y \tag{1}$$

is known to determine the asymptotic behavior of the nonlinear system of differential equations

$$dx/dt = A(t)x + f(t, x) \tag{2}$$

provided $f(t, x)$ is "sufficiently small." Our results, which are another contribution to this area, are motivated by two recent studies. Brauer and Wong [1] have obtained quite general results on the asymptotic relationships between the solutions of (1) and (2). We significantly weaken the hypotheses of one of their results; see Theorem 1 below. Toroshelidze [4] considered the problem of perturbing the asymptotic manifold (see definitions below) of a nonlinear scalar equation. This concept is discussed formally and in a more general setting by using systems (1) and (2); some related problems are also considered.

The techniques used in the proofs are a combination of the well-known comparison principle and the Schauder-Tychonoff fixed point theorem. Fundamental in the application of the comparison principle is a scalar equation

$$dr/dt = \omega(t, r). \tag{3}$$

In the above equations it will be assumed that $A(t)$ is a real valued continuous $n\times n$ matrix defined on the interval $J=[0, \infty)$; $f(t, x)$ is a real continuous n-vector valued function defined on $J\times R^n$ where R^n is Euclidean n-space; $\omega(t, r)$ is nonnegative and continuous on $J\times J$ with $\omega(t, r)$ nondecreasing in r, $r>0$, for each fixed $t\in J$. The fundamental matrix of (1) which is equal to the $n\times n$ identity matrix at $t=t_0$ will be designated by $Y(t)$. The symbol $|\cdot|$ will be used to denote any convenient vector norm.

The following theorem improves Theorem 3 of Brauer and Wong [1] by replacing a Lipschitz condition by a more general inequality.

[1] This research was supported in part by NSF grant GP-8921.

[2] This research was supported in part by NSF grant GP-9575.

This result is also closely related to a result of Onuchic [3, Theorem 2]. The proofs of the following results will appear elsewhere [2].

THEOREM 1. *Let* $\Delta(t)$ *be a nonsingular continuous matrix satisfying*

$$|\Delta(t)Y(t)| \leqq \alpha(t) \tag{4}$$

where $\alpha(t)$ *is a continuous positive function for* $t \geqq t_0 \geqq 0$. *Suppose also that* $f(t, x)$ *satisfies*

$$|Y^{-1}(t)f(t, x)| \leqq \omega(t, |\Delta(t)x|\alpha^{-1}(t)) \tag{5}$$

and that equation (3) *has a positive solution which is bounded on the interval* $t \geqq t_0$. *Given any solution* $y(t) = Y(t)c$ *of* (1) *with* $|c|$ *sufficiently small, there exists a solution* $x = x(t)$ *of* (2) *such that*

$$|\Delta(t)(x(t) - y(t))| = o(\alpha(t)), \qquad t \to \infty. \tag{6}$$

Theorem 1 of [1] and Theorem 1 above indicate that, under certain conditions, there is an asymptotic manifold of solutions of the perturbed equation (2) which is, in some sense, generated by equation (1). This comment motivates the following definitions.

DEFINITION 1. Let $\Delta(t)$, $\alpha(t)$ be as given in Theorem 1. The set $S = S(\Delta, \alpha)$ which consists of all solutions $x = x(t)$ of equation (2) that satisfy the order relation (6) for some constant vector c, will be called the *asymptotic manifold of* (2) *generated by* (1).

The next theorem is concerned with the following property of such manifolds.

DEFINITION 2. The set $S = S(\Delta, \alpha)$ is called *perturbable* if given any solution $x_0 = x_0(t)$ of (2) in S there exists a $\delta = \delta(t_0) > 0$ such that if $x(t)$ is a solution of (2) satisfying $|x(t_0) - x_0(t_0)| < \delta$ then $x(t)$ is also in S.

Definitions 1 and 2 are apparently new; however, for some special scalar equations, the next result has been anticipated by Toroshelidze [4].

THEOREM 2. *Let equation* (2) *have a unique solution to the initial value problem. Suppose that the hypotheses of Theorem* 1 *are satisfied and that* $\omega(t, r)r^{-1}$ *is nondecreasing in* r *for* $r > 0$ *and each fixed* t, $t \geqq t_0$. *If given any number* $r_\infty > 0$ *there exists a solution* $r(t)$ *of equation* (3) *such that* $\lim_{t\to\infty} r(t) = r_\infty$ *then the asymptotic manifold* $S(\Delta, \alpha)$ *of* (2) *generated by* (1) *is perturbable.*

The converse of this theorem is also valid for the special scalar equation (3) considered as a perturbation of the equation $dr/dt = 0$; however, in general the converse is not true even for first order scalar equations.

Finally, we state a result which is concerned with the perturbation of the set R of all solutions $y=y(t)$ of (1) for which there exists a solution $x=x(t)$ of (2) satisfying the asymptotic relationship (6).

THEOREM 3. *Let the hypotheses of Theorem* 1 *be satisfied. Furthermore, suppose that given any sufficiently large* T_0 *and positive constant* r_∞, *there exists a solution* $r=r(t)$ *of* (3) *which is valid for* $t \geqq T_0$ *and satisfies* $\lim_{t\to\infty} r(t)=r_\infty$. *Then, the manifold* R *is perturbable.*

Actually, under the hypotheses of Theorem 3, the conclusion of Theorem 1 holds for all initial vectors c in R^n. This result is consistent with the above mentioned result of Onuchic [3].

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