A METHOD OF ASCENT FOR SOLVING BOUNDARY VALUE PROBLEMS

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Stefan Bergman [1] and Ilya Vekua [4] have given representation formulas for solutions of the partial differential equation (1). We obtain an improvement of their results for the case of two independent variables (namely equation (2) with n set equal to 2). Furthermore, we are able to extend our result to higher dimensions (the *ascent*) by a remarkably simple variation of this two dimensional formula. Our representation (2) also contains Vekua's formulas [4, p. 59], for the Helmholtz equation in $n \geqq 2$ variables.

THEOREM 1. *Let $B(r^2)$ be an entire function of r^2, and $R(\zeta, \zeta^*; z, z^*)$ be the Riemann function*[1] *of the elliptic partial differential equation,*[2]

$$\Delta_2 u + B(r^2)u = 0, \quad r = \|x\|, \quad x = (x_1, x_2). \tag{1}$$

Then the function defined by

$$u(x) = h(x) + \int_0^1 \sigma^{n-1} G(r; 1 - \sigma^2) h(x\sigma^2) d\sigma, \quad x = (x_1 \cdots, x_n) \tag{2}$$

where $h(x)$ is harmonic in a star-like region (with respect to the origin) D, and $G(r, 1-\sigma^2) \equiv -2rR_1(r\sigma^2, 0; r, r)$, is a solution of

$$\Delta_n u + B(r^2)u = 0, \tag{3}$$

for $x \in D$. Furthermore, each regular solution of (3) *may be represented in the form* (2).

PROOF. Using Bergman's integral operator of the first kind [1, p. 10], which generates a complete system of solutions for equation (1), namely

$$u(x) = 2 \operatorname{Re}\left\{ \int_0^{+1} E(r, t) f(z[1 - t^2]) \frac{dt}{(1 - t^2)^{1/2}} \right\}, \quad \|x\| = r \tag{4}$$

one may obtain the alternate representation,

$$u(x) = h(x) + \sum_{l \geq 1} 2 \frac{e_l(r^2)}{B(l, \frac{1}{2})} \int_0^1 \sigma(1 - \sigma^2)^{l-1} h(\sigma^2 x) d\sigma, \tag{5}$$

[1] See [2, Chapter V], [3, Chapter III], and [4].

[2] $\Delta_n \equiv \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \cdots + \partial^2/\partial x_n^2$, and $z = x_1 + ix_2$, $z^* = x_1 - ix_2$, $\zeta = \xi + i\eta$, $\zeta^* = \xi - i\eta$.

where

$$h(\mathbf{x}) \equiv 2\int_0^1 \operatorname{Re}\{f(z[1-t^2])\}\frac{dt}{(1-t^2)^{1/2}}, \tag{6}$$

and

$$e_l(r^2) = \frac{1}{(2l)!}\left[\frac{\partial^{2l}}{\partial t^{2l}}E(r,t)\right]_{t=0}. \tag{7}$$

Since (4) may be seen to be the real part of a solution of a complex Goursat problem (see [3, Chapter III]), it may also be represented in terms of the Riemann function of the formally hyperbolic equation which arises from substituting $z=x_1+ix_2$, $z^*=x_1-ix_2$ into (1). Such a substitution is clearly valid since $B(zz^*)$ is holomorphic in C^2. By identifying the real part of the Riemann function representation of this Goursat problem with (5) one obtains equation (2) with $n=2$. In order to establish (2) for $n>2$ one must obtain first a generalization of (4) for $n>2$. This may be done by a representation of the form

$$u(\mathbf{x}) = \int_0^1 t^{n-2}E(r,t;n)H(\mathbf{x}[1-t^2])\frac{dt}{(1-t^2)^{1/2}}, \tag{8}$$

where $E(r, t; n)$ is a solution of

$$(1-t^2)E_{rt} + (n-3)(t^{-1}-t)E_r + rt\left(E_{rr} + \frac{n-2}{r}E_r + BE\right) = 0, \tag{9}$$

which satisfies

$$\lim_{t\to 0^+}(t^{n-3}E_r)r^{-1} = 0, \qquad \lim_{t\to 1^-}((1-t^2)^{1/2}E_r)r^{-1} = 0, \qquad \lim_{r\to 0^+} E = 0. \tag{10}$$

It may be shown using the method of majorants that such solutions exist. Representation (8) may be reformulated as

$$u(\mathbf{x}) = h(\mathbf{x}) + \sum_{l\geq 1} c_l(r^2;n)\int_0^1 \sigma^{n-1}(1-\sigma^2)^{l-1}h(\mathbf{x}\sigma^2)\,d\sigma, \tag{11}$$

with

$$c_l(r^2;n) = 2e_l(r^2;n)\Gamma\left(l+\frac{n-1}{2}\right)\left\{\Gamma\left(\frac{n-1}{2}\right)\Gamma(n)\right\}^{-1}, \tag{12}$$

and where $e_l(r^2; n)$ are the *even* Taylor coefficients of $E(r; t; n)$. The function $G(r, \tau)$ defined by

$$(13) \qquad G(r, \tau) \equiv \sum_{l\geq 1} c_l(r^2; n)\tau^{l-1}$$

is seen to satisfy the partial differential equation,

$$(14) \qquad 2(1-\tau)G_{r\tau} - G_r + r(G_{rr} + BG) = 0,$$

and the data

$$G(0, \tau) = 0, \qquad G(r, 0) = -\int_0^r rB(r^2)dr,$$

and is therefore *independent of the dimension n*. This proves the first part of our theorem. To realize that each solution may be written in this form we need only recognize that (2) may be rewritten as a Volterra equation by a simple change of variables.

ADDED IN PROOF. It also may be shown that (2) has the following inverse: $h(\mathbf{x}) = u(\mathbf{x}) + \int_0^1 \sigma^{n-1} g(r, 1-\sigma^2)u(\mathbf{x}\sigma^2)d\sigma$, where $g(r; 1-\sigma^2) = -2\hat{R}_1(r\sigma^2, 0; r, r)$, and $\hat{R}$ is the Riemann function of (1) with B replaced by minus B.

REMARK. For the special case of *Helmholtz's equation* Vekua has already given such a method of ascent. Namely for $B(r^2) \equiv \lambda^2$ one has [4, p. 59]

$$(15) \quad u(\mathbf{x}) = h(\mathbf{x}) - xr\int_0^1 \sigma^{n-1} J_1(\lambda r(1-\sigma^2)^{1/2})h(\mathbf{x}\sigma^2)\frac{d\sigma}{(1-\sigma^2)^{1/2}},$$

holding for integer $n \geqq 2$. Our representation (2) contains this example as a special case, which may be seen by a simple computation.

THEOREM 2. *Let D be star-like with respect to the origin and $B(r^2)$ an entire function, such that $B(r^2) < 0$ in D. Furthermore, let ∂D be a Lyapunov boundary and $f(\mathbf{x})$ a continuous function on ∂D. Then there exists a unique solution of* (3) *which may be represented as in equation* (2), *where $h(\mathbf{x})$ is given as a double layer potential*

$$(16) \qquad h(\mathbf{x}) = \frac{\Gamma(n/2)}{\pi^{n/2}}\int_{\partial D}\mu(y)\frac{\partial}{\partial \nu_y}\left(\frac{1}{\|\mathbf{x}-y\|^{n-2}}\right)d\omega_y,$$

and $\mu(y)$ is a solution of the Fredholm integral equation,

$$f(\mathbf{x}) = \mu(\mathbf{x}) + \frac{\Gamma(n/2)}{\pi^{n/2}} \int_{\partial D} \mu(y) \left\{ \frac{\partial}{\partial \nu_y} \left(\frac{1}{\|\mathbf{x} - y\|^{n-2}} \right.\right.$$

$$\left.\left. - 2r \int_0^1 \sigma^{n-1} R_1(r\sigma^2, 0; r, r) \frac{\partial}{\partial \nu_y} \left(\frac{1}{\|\mathbf{x}\sigma^2 - y\|^{n-2}} \right) \right) \right\} d\omega_y, \tag{17}$$

$$\mathbf{x} \in \partial D.$$

PROOF. This follows by substituting the double layer potential into the representation (11), using the Fubini theorem to change orders of integration, and computing the residue as $\mathbf{x} \to \partial D$ from the inside.

REFERENCES

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