CERTAIN MAPPINGS OR DECOMPOSITIONS WHICH ARE TOPOLOGICALLY PROJECTIONS

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Introduction. A general question which is of interest is the following. Suppose that f is a mapping of a compact metric continuum X onto a metric space Y. *Under what conditions is there an embedding of X and Y in E^n (Euclidean n-space) or H^ω (Hilbert space) so that f is topologically equivalent to a projection onto Y defined by some collection of parallel hyperplanes?* Theorem 1 below provides an answer for a very special case of this general question. Although this theorem is actually a corollary of a more general theorem, we feel that its proof provides motivation and understanding for the main theorem.

THEOREM 1. *Suppose that U is the Universal 1-dimensional Menger Curve* [1] *and that f is a light open mapping of U onto I (the interval* [0, 1]) *such that $f^{-1}(x)$ is homeomorphic to a Cantor set for each x in I. Then there is a homeomorphism h of U into E^3 such that the mapping p defined by projecting U onto I through planes parallel to the yz-plane is topologically equivalent to f, that is, $ph=f$.*

We shall sketch a proof of this theorem. Our proof depends on an important theorem of J. H. Roberts [5] concerning contractibility in spaces of homeomorphisms, some very useful techniques of Dyer and Hamstrom [2], and a powerful selection theorem of E. A. Michael [4].

Statements of some results used in our proofs. Suppose that X is a compact metric space and dimension $X=n$ (an integer). For each positive integer k, let $H(X, I^k)$ be the space of all homeomorphisms of X into I^k (a k-cell) and let $C(X, I^k)$ be the space of all mappings of X into I^k. The metric, in each case, is the usual one: $\rho(f, g) = \max d(f(x), g(x))$ for x in X and d is the usual metric for I^k.

THEOREM (J. H. ROBERTS [5]). *Suppose that each of X and K is a compact metric space,* $\dim X=n$, $\dim K=r$, *and* $k \geqq 2n+2+r$. *Let α_0 and α_1 be mappings of K into $C(X, I^k)$. Then there exists a homotopy $f: K\times I \to C(X, I^k)$ such that*

(1) $f(\omega, 0)=\alpha_0(\omega)$, $f(\omega, 1)=\alpha_1(\omega)$, $\omega \in K$, *and*

(2) *for each t,* $0<t<1$, $f(\omega, t) \in H(X, I^k)$.

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THEOREM (E. A. MICHAEL [4]). *If each of A and B is a metric space, A is complete, covering dimension of $B \leqq n+1$, Z is a closed subset of B, F is a function taking A onto B such that the collection of inverses under F is lower semicontinuous (defined below) and equi-LC^n (as defined below), and f is a mapping of Z into A such that for z in Z, $f(z) \in F^{-1}(z)$, then there is a neighborhood U of Z in B such that f can be extended to a mapping f^* of U into A such that for $b \in U, f^*(b) \in F^{-1}(b)$. If each inverse under F has the property that its homotopy groups of order $\leqq n$ vanish, then U may be taken to be the space B.*

Notation and definitions. In this paper, all *mappings* are *continuous* and all *spaces* are *metric*. A mapping f of a space X into a space Y is *light* iff $f^{-1}f(x)$ is totally disconnected for each x in X. And, f is *open* iff for each U open in X, $f(U)$ is open relative to $f(X)$. A characterization of the Universal 1-dimensional Curve U may be found in R. D. Anderson's paper [1].

DEFINITION (DYER AND HAMSTROM [2]). A mapping $p: T \rightarrow B$ is said to be *completely regular* iff for each $\epsilon > 0$ and each point b in B, there is a $\delta > 0$ such that if $x \in B$ and $d(x, b) < \delta$, then there exists a homeomorphism h_{bx} of $p^{-1}(b)$ onto $p^{-1}(x)$ which moves no point as much as ϵ.

DEFINITION. A collection G of closed point sets filling a metric space X (i.e., the union of the elements of G is X) is said to be equi-LC^n iff for each $\epsilon > 0$, g in G, and $x \in g$, there is a $\delta > 0$ such that if $h \in G$ and f is a mapping of a k-sphere S^k, $0 \leqq k \leqq n$ into $h \cap N_\delta(x)$, then there is an extension F of f to the $(k+1)$-disk D^{k+1}, into $h \cap N_\epsilon(x)$.

The hypothesis of Theorem 1 is not vacuously satisfied. Such mappings are easy to construct.

Indication of a proof of Theorem 1. Let A denote a unit cube (3-cell) in E^3 whose vertices are (0, 0, 0), (0, 0, 1), (0, 1, 0), (1, 0, 0), (1, 0, 1), (0, 1, 1), (1, 1, 0), and (1, 1, 1). Let C_x denote a 2-cell section of A cut by the plane perpendicular to the x-axis at x.

For each x, let $H(f^{-1}(x), C_x)$ denote the space of all homeomorphisms of $f^{-1}(x)$ into C_x. For convenience, we shorten this to H_x. We use the usual metric on H_x, i.e., for g, h in H_x, $\rho(g, h) = \max \{\rho[g(x), H(x)]\}$. Now, H_x is a topologically complete metric space.

Consider the collection H of all H_x and let H^* denote the union of the elements of H. The space H^* is a topologically complete metric space. This follows from a theorem in [3]. However, we shall indicate here how a metric may be defined.

A metric for H^*. If $g \in H^*$, then $g \in H_x$ for some x. Let $\hat{g}$ denote the graph of g in $U\times C$ where C is a 2-cell. For each pair of elements g, h of H^* where $g\in H_x$ and $h\in H_y$, let $D(g, h)=$ Hausdorff distance between $\hat{g}$ and $\hat{h}$. Although D is a metric for H^*, it may not be complete. However, H^* is a topologically complete metric space. This follows from Theorem 1 of [3].

By a theorem of Roberts [5], H_x for each x is locally connected. The collection H of all H_x is equi-locally connected in the homotopy sense (equi-LC^0). That is, for each H_x, $p\in H_x$, and $\epsilon>0$, there is a $\delta>0$ such that if ϕ is a mapping of S^0 (a 0-sphere or pair of points) into $N_\delta(p)\cap H_y$ for H_y in H, then ϕ can be extended to a mapping Φ which takes a 1-cell into $N_\epsilon(p)\cap H_y$. This may be proved by first showing that f is actually completely regular. Next, apply an argument similar to Dyer and Hamstrom [2] or to mine in [3].

Let H^* be the union of the elements of H and F denote the function from H^* onto $I=[0, 1]$ such that $F^{-1}(x)=H_x$. It follows that F is lower semicontinuous. That is, if $\{h_i\}\to h$ where h_i, $h\in H$, then H_h is in the closure of $\bigcup_{i=1}^{\infty} H_{h_i}$. See [3, p. 137]. Now by a selection theorem of Michael [4], there is a continuous selection Φ from an open interval (a, b) to H^* such that $\Phi(x)\in F^{-1}(x)=H_x$. By Corollary 2 of [5], $F^{-1}(x)$ for each x in I is arcwise connected. Thus, by Michael's Theorem [4, p. 563], (a, b) may be taken as the space $[0, 1]$. The mapping Φ induces a homeomorphism h from $f^{-1}[0, 1]=U$ into A (a 3-cell) such that $h|f^{-1}(x)=\Phi(x)$. That is, for u in U, $h(u)=\Phi[f(u)](u)$. It follows that $f=ph$ where p is the projection of A onto I by planes parallel to the yz-plane. The theorem is proved.

REMARKS. Projections need not be local products (locally trivial fiber spaces), even in the case that $p:X\to Y$ has the property that all sets $p^{-1}p(x)$ are homeomorphic for the various $x\in X$, X is a Peano continuum, p is open, and p is monotone. See Ungar's example [6].

Main theorem. Now, we are ready to state the general theorem for which Theorem 1 is a special case.

THEOREM 2. *Suppose that f: $X\Rightarrow I^{r+1}$ is a completely regular mapping, X is a complete metric space, for each x in X, $f^{-1}f(x)\cong K$, a compact n-dimensional set. Let $k\geqq 2n+2+r$. Then there is a homeomorphism h of X into I^{k+r+1} such that $f=ph$ where p is the projection mapping of $I^k\times I^{r+1}$ onto I^{r+1}.*

It should be clear from the indicated proof of Theorem 1 that a similar argument yields Theorem 2.

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