ON THE CONSTRUCTION OF THE EILENBERG-MOORE SPECTRAL SEQUENCE

BY LARRY SMITH[1]

Communicated by Frank Peterson, January 13, 1969

Suppose that

$$\mathfrak{F} \qquad \begin{array}{ccc} E & \longrightarrow & E_0 \\ \downarrow & & \downarrow \\ B & \longrightarrow & B_0 \end{array}$$

is a fibre square. Under suitable regularity conditions Eilenberg and Moore have introduced [6] (see also [16], [17]) a spectral sequence $\{E_r(\mathfrak{F}), d_r(\mathfrak{F})\}$ with

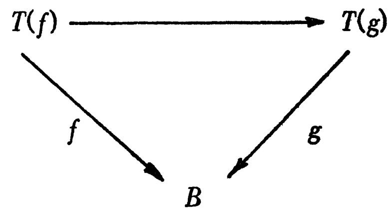
$$E_r(\mathfrak{F}) \Rightarrow H^*(E; k),$$
$$E_2(\mathfrak{F}) = \operatorname{Tor}_{H^*(B_0;k)}(H^*(B;k), H^*(E_0; k)),$$

where $H^*(\ ; k)$ denotes cohomology with coefficients in the field k. This spectral sequence has proved to be a powerful tool for dealing with numerous problems in algebraic topology [3], [16], [17]. The constructions of [6] (also [16]), are entirely algebraic in nature and have left open several rather important points, among them the relation between the spectral sequence and the Steenrod algebra when $k = Z_p$, p a prime. (That such a relation should exist is strongly suggested by [16, §4] and [9], and has provided a large portion of the motivation for the present work.) We will remedy this shortcoming by providing a geometric construction of the spectral sequence, that will also clarify the status of the spectral sequence for generalized cohomology theories.

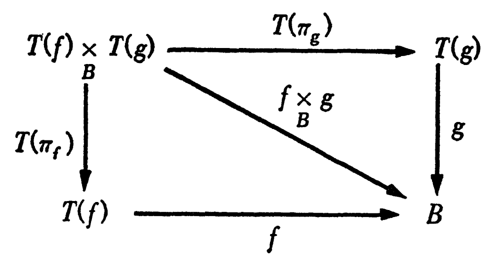
We will adopt the point of view, first employed by Hodgkin in [7] that the Eilenberg-Moore spectral sequence should be the Künneth spectral sequence for a suitable cohomology theory on a suitable category. This point of view contains a good deal more than the germ of the construction. For there already exists a method, geometric in nature, for dealing with Künneth type problems for general cohomology theories [2]. This method of Atiyah has recently come under rescrutiny and has been extended to more general situations [1; I], [5; §8], [7]. This suggests that we try to adopt Atiyah's method to the case at hand.

[1] During the course of this work the author was an Air Force Office of Scientific Research Postdoctoral Fellow.

With this in mind, let us fix throughout the remainder of this discussion a topological space B. We introduce the category Top/B of topological spaces over B. An object of Top/B is a map $f\colon T(f)\to B$; $T(f)$ denotes the total space of f. A morphism $\alpha\colon f\to g$ is a commutative triangle



Note that the category Top/B has products; the familiar fibre product. If f, $g\in$obj Top/B, then the diagram



defines the product $f\times_B g$ of f and g in Top/B. There is also a base-point (=terminal object), 1_B, given by the identity map $1_B\colon B\to B$.

The category Top/B has many of the properties of the category of spaces, and it is this that we will exploit. This point of view is due to Becker and Milgram [4], James [8], Meyer [11] and McClendon [10].

Suppose now that $\mathcal{H}^*$ is an unreduced cohomology theory on the category of spaces. We may prolong $\mathcal{H}^*$ to a sort of cohomology theory, $\mathcal{H}_B^*$, on Top/B by setting

$$\mathcal{H}_B^*(f) = \mathcal{H}^*(T(f)).$$

The coefficients of this cohomology theory ought to be $\mathcal{H}_B^*(1_B) = \mathcal{H}^*(B)$. Thus the Künneth theorem for $\mathcal{H}_B^*$, *on* Top/B, ought to take the form of a spectral sequence $\{E_r(f, g), d_r(f, g)\}$ with

$$E_r(f, g) \Rightarrow \mathcal{H}_B^*(f\times_B g) = \mathcal{H}^*(Tf\times_B Tg),$$

$$E_2(f, g) = \mathrm{Tor}_{\mathcal{H}_B^*(1_B)}(\mathcal{H}_B^*(f), \mathcal{H}_B^*(g)) = \mathrm{Tor}_{\mathcal{H}^*(B)}(\mathcal{H}^*(Tf), \mathcal{H}^*(Tg)),$$

which is exactly what one would expect of an Eilenberg-Moore type spectral sequence for the cohomology theory $\mathcal{H}^*$.

A moment's reflection on Atiyah's method for dealing with Künneth type problems shows that it is homotopy theoretic in nature. Homotopy theory invariably requires a basepoint, and so we will be forced to work in the category, $(\mathrm{Top}/B)_*$ of pointed spaces over B. An object of $(\mathrm{Top}/B)_*$ is a pair (f, s) where $f: T(f)\to B$ and $s: B\to T(f)$ is a cross-section to f. Morphisms in $(\mathrm{Top}/B)_*$ must preserve cross-sections. The category of pointed spaces over B enjoys all the ingredients for doing homotopy theory. We have the fibrewise construction of mapping cones, mapping cylinders, suspensions etc. [4], [8], [10], [11]. It is therefore clear that we may speak of half exact functors and cohomology theories on $(\mathrm{Top}/B)_*$.

There is a functor sending pointed spaces to pointed spaces over B given by

$$(X, x_0) \mapsto X \times B \xrightarrow[P_B]{} B, \qquad s: B \to X \times B$$

$$P_B(x, b) = b \qquad s(b) = (x_0, b).$$

Thus we have the spheres over B, (S_B^n, s_n).

Consider now a reduced multiplicative cohomology theory *on the category* $(\mathrm{Top/B})_*$ $\mathcal{H}^*$. Let us suppose:

(1) for any $(g,s)\in \mathrm{obj}\,(\mathrm{Top/B})_*$ there exists a morphism $\alpha:(g,s)\to(h,t)$ such that

(a) $\mathcal{H}^*(\alpha): \mathcal{H}^*(h, t)\to\mathcal{H}^*(g, s)$ is surjective,

(b) $\mathcal{H}^*(h, t)$ is a projective $\mathcal{H}^*(S_B^0, s_0)$-module;

(2) if in (1), g is a fibration we may choose α so that h is also a fibration;

(3) if in (1) or (2), $\mathcal{H}^*(g, s)$ has finite type we may choose α so that $\mathcal{H}^*(h, t)$ has finite type;

(4) if $(h, t)\in(\mathrm{Top}/B)_*$ with $\mathcal{H}^*(h, t)$ a projective $\mathcal{H}^*(S_B^0, s_0)$-module of finite type, then the exterior product

$$\mathcal{H}^*(-, -) \otimes_{\mathcal{H}^*(S_B^0, s_0)} \mathcal{H}^*(h, t) \to \mathcal{H}^*(- \wedge_B h, - \wedge_B t)$$

is an isomorphism of functors whenever h is a fibration. (Here $\wedge_B$ is the smash product over B.)

Suppose now that $(g, s)\in(\mathrm{Top}/B)_*$. We may iterate the construction (1) to obtain a sequence

$$(g, s) \to (h_0, t_0) \to (h_{-1}, t_{-1}) \to \cdots$$

of pointed spaces over B, in which successive maps are cofibrations in (Top/B)$_*$. For any $(f, r) \in$ obj (Top/B)$_*$ we may thus obtain a filtered space over B by applying the Puppe sequence construction,

$$(f \underset{B}{\wedge} g_{-n}, r \underset{B}{\wedge} s_{-n}) \to (f \underset{B}{\wedge} S_B g_{-n+1}, r \underset{B}{\wedge} S_B g_{-n+1}) \to \cdots$$

$$\to (f \underset{B}{\wedge} S_B^n g, r \underset{B}{\wedge} S_B^n s)$$

where n is a positive integer and S_B^i denotes the ith-iterate of the suspension functor on (Top/B)$_*$. The spectral sequence of this filtered object has

$$E_r \Rightarrow \mathcal{H}^*(f \underset{B}{\wedge} S_B^n g, r \underset{B}{\wedge} S_B^n s) = s^n + \mathcal{H}^*(f \underset{B}{\wedge} g, r \underset{B}{\wedge} s)$$

and if g is a fibration with $\mathcal{H}^*(g, s)$ of finite type, then

$$E_2^{s,t} = \operatorname{Tor}^{s,t+n}_{\mathcal{H}^*(S_B^0, s_0)} \mathcal{H}^*(f, r), \mathcal{H}^*(g, s)$$

for $s > -n$. Allowing $s \to \infty$ we obtain increasing portions of the desired Künneth spectral sequence and a convergence problem. The situation is analogous to the construction of the Adams spectral sequence. One way around the convergence problem is to assume

(5) for any $(f, s) \in \mathrm{Top}_*/B$, $\mathcal{H}^j(f, s) = 0$ for $j < 0$.

We then obtain:

THEOREM. *Let $\mathcal{H}^*$ be a cohomology theory on* (Top/B_*) *satisfying* (1)–(5). *For each* $(f, r), (g, s) \in$ obj (Top/B_*), *with $\mathcal{H}^*(g, s)$ of finite type and g a fibration, there exists a spectral sequence*

$$\{E_r((f, r), (g, s)), d_r((f, r), (g, s))\}$$

with

$$E_r((f, r), (g, s)) \Rightarrow \mathcal{H}^*(f \underset{B}{\wedge} g, r \underset{B}{\wedge} s),$$

$$E_2((f, r), (g, s)) = \operatorname{Tor}_{\mathcal{H}^*(S_B^0, s_0)}(\mathcal{H}^*(f, r), \mathcal{H}^*(g, s)).$$

The Künneth spectral sequence for unreduced cohomology theories on Top/B now follows by the usual trick of adjoining a basepoint to pass to smash products in Top/B_*. Note that (1)–(5) are satisfied when B is simply connected and $\mathcal{H}_B$ is given by

$$(f, s) \to H^*(T(f), s(B); Z_p),$$

p a prime. As a consequence of the geometric nature of the construction we obtain

THEOREM. *Let p be a prime and B a simply connected space with $H^*(B; \mathbf{Z}_p)$ of finite type. Suppose given $f, g \in \operatorname{obj} \mathrm{Top}/B$, with g a fibration and $H^*(Tg; \mathbf{Z}_p)$ of finite type. Then there exists a spectral sequence $\{\mathbf{E}_r(f, g), d_r(f, g)\}$ of algebras with*

(1) $\mathbf{E}_r(f, g) \Rightarrow H^*(Tf \times_B Tg; \mathbf{Z}_p)$,

(2) $\mathbf{E}_2(f, g) = \mathrm{Tor}_{H^*(B; \mathbf{Z}_p)} (H^*(Tg; \mathbf{Z}_p), H^*(Tf; \mathbf{Z}_p))$.

Moreover, for each $r \geqq 2$, $\mathbf{E}_r^{s,*}(f, g)$ is an $\mathfrak{A}^*(p)$-module ($\mathfrak{A}^*(p)$ denotes the mod p Steenrod algebra) and

(3) the derivations

$$d_r(f, g): \mathbf{E}_r^{s,*}(f, g) \to \mathbf{E}_r^{s+r,*}(f, g)$$

are morphisms of $\mathfrak{A}^*(p)$-modules of degree $1 - r$;

(4) under the identification (2) the $\mathfrak{A}^*(p)$-modules structure coincides with the standard structure [9] on

$$\mathrm{Tor}^{s,*}_{H^*(B; Z_p)}(H^*(Tf; \mathbf{Z}_p), H^*(Tg; \mathbf{Z}_p));$$

(5) the convergence in (1) is as $\mathfrak{A}^*(p)$-modules; i.e.,

$$F^{-s}H^*(Tf \times_B Tg; \mathbf{Z}_p) \subset H^*(Tf \times_B Tg; \mathbf{Z}_p)$$

is an $\mathfrak{A}^*(p)$-module and

$$\mathbf{E}_0^{s,*}H^*(Tf \times_B Tg; \mathbf{Z}_p) = \mathbf{E}_\infty^{s,*}(f, g)$$

as $\mathfrak{A}^*(p)$-modules;

(6) the multiplication

$$\mathbf{E}_r^{s',*}(f, g) \otimes \mathbf{E}_r^{s'',*}(f, g) \to \mathbf{E}_r^{s'+s'',*}(f, g)$$

is a morphism of $\mathfrak{A}^*(p)$-modules.

A result similar to this has also been obtained by D. Rector [**14**], and in a special case by V. Puppe [**13**], by semisimplicial methods. Additional results and applications will appear elsewhere. In particular we will discuss the convergence problem in more detail. The geometric nature of the construction leads to Eilenberg-Moore type spectral sequences for certain generalized homology and cohomology theories. These two will be discussed elsewhere [**17**].

Note that the spectral sequence of [**12**], [**15**] may be obtained by a similar philosophy applied to the category of G-spaces.

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Institut des Hautes Études Scientifiques, Bures sur Yvette, France
University of Virginia, Charlottsville, Virginia