RESEARCH ANNOUNCEMENTS

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ON PROJECTIONS OF SELFADJOINT OPERATORS AND OPERATOR PRODUCT ADJOINTS

BY KARL GUSTAFSON[1]

Introduction. The proofs given in §§1 and 2 of this note were contained in a letter from the present writer to G. Lumer, sent shortly after the appearance of [1]. The subsequent appearance of [2] makes it desirable to publish these demonstrations. In [3] we mention an observation relevant to the topic of [2].

1. On [1]. [1] consists primarily of the proof of the following lemma and an addendum. Let T be any selfadjoint operator in $\mathcal{3C}$, a Hilbert space, $\mathcal{Y}$ a closed subspace in $\mathcal{3C}$, P the orthogonal projection of $\mathcal{3C}$ onto $\mathcal{Y}$, $\chi = \mathcal{Y}^\perp$, the orthogonal complement of $\mathcal{Y}$ in $\mathcal{3C}$. Let $T_0 = PTP$.

LEMMA 1. *If χ is finite dimensional, then T_0 is selfadjoint.*

OUR PROOF. $\mathfrak{D}(T_0) = \mathfrak{D}(TP)$ is dense by [3, Theorem IV 2.7(iv), p. 103; note $\mathfrak{D}(T)$ dense is the only property of T used in (iv)]. Any time $\mathfrak{D}(TP)$ is dense, $(T_0)^* \equiv [P(TP)]^* = (TP)^* P \supseteq PTP = T_0$, and T_0 is selfadjoint iff $(TP)^* = PT$ iff PT is closed, the latter implication seen from $(TP)^* \equiv (T^*P^*)^* = (PT)^{**} \supseteq PT$, equality holding iff PT is closed. But PT is closed by [3, Theorem IV 2.7(i)].

The addendum of [1] asserts that a remark credited to G. Lumer in the acknowledgment of [4] is incorrect. However, the remark is correct in the context of *bounded* operators, which clearly was the context intended by Williams [4]. In that context, $T_0 = ATA$ is obviously selfadjoint whenever A and T are.

2. On [2]. Let S and T be densely defined closed linear operators in $\mathcal{3C}$; [2] deals with the interesting question of when $(TS)^* = S^*T^*$

is true beyond the well-known case where T is bounded and every-where defined, and adopting the ideas of [1] the following results are stated (proofs deferred to a detailed paper to be submitted else-where).

THEOREM. *If T is a closed domain dense linear operator on $\mathfrak{IC}$, and S is a bounded everywhere defined linear operator whose image is a closed subspace of finite codimension in $\mathfrak{IC}$, then $(TS)^* = S^*T^*$.*

COROLLARY. *If S and T are selfadjoint, and if S is bounded, has a closed image, and has finite dimensional kernel, then STS is selfadjoint.*

OUR PROOFS. The corollary is a special case of the theorem; alter-nately, replace P by S in the above proof. For the theorem $\mathfrak{D}(TS)$ is dense as above, and $(TS)^* \equiv (T^{**}S^{**})^* = [(S^*T^*)^*]^* \equiv (S^*T^*)^{**} \supseteq S^*T^*$, equality holding iff S^*T^* is closed. But the latter is true by [3, Theorem IV 2.7(i)].

3. **On when** $(TS)^* = S^*T^*$. It should be observed, without further elaboration, that the question of [2] has a fairly general answer via the Fredholm theory (e.g., see [3]).

PROPOSITION 1. *If T and S are Fredholm operators, then $(TS)^* = S^*T^*$.*

PROOF. Let κ denote the index of an operator. Since $(TS)^* \supseteq S^*T^*$, it is sufficient to demonstrate that $\kappa((TS)^*) = \kappa(S^*T^*)$. By the Fredholm theory, one has $\kappa((TS)^*) = -\kappa(TS) = -\kappa(T) - \kappa(S) = \kappa(T^*) + \kappa(S^*) = \kappa(S^*T^*)$.

ADDITIONAL REMARK. We have found that Proposition 1, with T just closed, was shown in [5]; see also [6, Lemma 2.3]. Accordingly we state:

PROPOSITION 2. *Let T and S be densely defined linear operators in a Hilbert space. If S is closed and $R(S)$ has finite codimension, then $(TS)^* = S^*T^*$.*

PROOF. As in [5, Lemma 4.1]. More specifically: T closed is not needed (e.g., by referring to [3]); $N(S)$ and $R(S)$ closed are implied by the hypotheses; dim $N(S)$ need not be finite; and the bound [5, (4.3)] holds when S and $R(S)$ are closed.

Further ramifications are conceivable (e.g., assume a little more on T, a little less on S), as well as extensions to normed linear spaces, etc. On the other hand, Proposition 2 is rather sharp for arbitrary T in that the conditions on S are those generally required both to conclude that $(TS)^*$ exists and that S^*T^* is closed.

References

1. W. Stenger, *On the projection of a selfadjoint operator*, Bull. Amer. Math. Soc. 74 (1968), 369–372.

2. S. Holland, Jr., *On the adjoint of the product of operators*, Bull. Amer. Math. Soc. 74 (1968), 931–932.

3. S. Goldberg, *Unbounded linear operators*, McGraw-Hill, New York, 1966.

4. J. P. Williams, *Spectra of products and numerical ranges*, J. Math. Anal. Appl. 17 (1967), 214–220.

5. S. Kaniel and M. Schechter, *Spectral theory for Fredholm operators*, Comm. Pure Appl. Math. 16 (1963), 423–448.

6. H. O. Cordes and J. P. Labrousse, *The invariance of the index in the metric space of closed operators*, J. Math. Mech. 12 (1963), 693–720.

UNIVERSITY OF COLORADO, BOULDER, COLORADO 80302