Linear and quasilinear elliptic equations, by O. A. Ladyženskaja and N. N. Ural'ceva. Translated by Scripta Technica, Math. in Science and Engineering, vol. 46, Academic Press, New York, 1968. xviii +495 pp. $24.00.

The present book could be aptly subtitled: "How to choose the right test function" for it is an outstanding treatise devoted mainly to estimates achieved through intricate choices of test functions in the weak or integral forms of the various elliptic equations under consideration. The results and methods are largely due to the authors, although many of their techniques have roots in the classical works of Bernstein and the more recent work of De Giorgi. The book is concerned with the following facets of the theory of second order, linear and quasilinear elliptic equations:

(i) the solvability of the Dirichlet problem in various both classical and generalized formulations;

(ii) interior estimates of solutions and their derivatives;

(iii) global estimates of solutions and their derivatives in bounded domains;

(iv) interior regularity of solutions;

(v) global regularity of solutions in bounded domains.

Naturally, these themes are interwoven. The derivation of estimates can generally be adapted so as to provide regularity results. The solvability of the Dirichlet problem is reduced to a question of a priori global estimates through the application of topological fixed point theorems in the appropriate Banach spaces.

The types of second order equations examined are distinguished according to whether or not they are (a) linear or (b) of divergence structure. Accordingly four different types are treated.

$$(1)\qquad Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left[a_{ij}(x)u_{x_j} + a_i(x)u\right] + \sum_{i=1}^{n} b_i(x)u_{x_i} + a(x)u = f - \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}.$$

$$(2)\qquad Lu = \sum_{i,j=1}^{n} a_{ij}(x)u_{x_ix_j} + \sum_{i=0}^{n} a_i(x)u_{x_i} + a(x)u = f(x).$$

$$(3)\qquad \sum_{i=1}^{n} \frac{d}{dx_i}\left[a_i(x, u, u_x)\right] + a(x, u, u_x) = 0.$$

$$(4)\qquad \sum_{i,j=1}^{n} a_{ij}(x, u, u_x)u_{x_ix_j} + a(x, u, u_x) = 0.$$

The general second order, quasilinear equation (4) is of *divergence form* if it may be written in the form (3). Equations (1) and (3) are considered both with and not necessarily with smooth coefficients. Throughout most of the work the equations are assumed to be *uniformly elliptic*, i.e., in (1) and (2)

$$\nu|\xi|^2 \leqq a_{ij}(x)\xi_i\xi_j \leqq \nu^{-1}|\xi|^2$$

for all vectors ξ in R^n and for some positive ν; whilst in (3)

$$\nu(|u|)(1+|p|)^{m-2}|\xi|^2 \leqq \frac{\partial a_i}{\partial p_j}(x,u,p)\xi_i\xi_j \leqq \mu(|u|)(1+|p|)^{m-2}$$

where ν and μ are positive, respectively nonincreasing and nondecreasing functions and $m>1$; and finally in (4)

$$\nu(|u|)|\xi|^2 \leqq a_{ij}(x,u,p)\xi_i\xi_j \leqq \mu(|u|)|\xi|^2.$$

The book is divided into fourteen chapters. Chapter 1 is a useful introduction and guide to the work. Here the various growth and integrability conditions to be imposed on the coefficients of the equations (in addition to uniform ellipticity) are motivated by means of specific examples. A guiding criterion is that solutions should be unique in the small. Chapter 2 establishes several lemmas upon which subsequent estimates are based. The lemmas refer to the Hölder and boundedness properties satisfied by functions in the classes B_m, B_m^N, which are characterized by certain integrodifferential inequalities.

The theory of linear equations is developed in Chapter 3. It centers about the solvability of boundary value problems (mainly the Dirichlet problem) in Hölder and W_2' spaces. The Schauder theory for equation (2) is presented in §§1–3. A W_2' theory of equations of the form (1) whose coefficients are not necessarily smooth is given in §§4–6—this is an extension of the functional analytic theory for linear equations with smooth coefficients due to Friedrichs and others. The remainder of the chapter deals with solvability of equation (1) in W_2^2, the De Giorgi-Nash results and their extensions to equations of form (1), diffraction problems and finally with the more classical, two variable theory.

Equations of divergence structure, i.e. equations of the form (3), are treated in Chapter 4. Interior and boundary derivative estimates are proved along with the regularity of W_m' solutions where m is the constant in the definition of uniform ellipticity mentioned earlier in this review. The Dirichlet problem is studied in the latter part of the chapter. The estimates established in Chapter 4 are also used in the following chapter to study the regularity of solutions of variational problems for multiple integrals of the form $\int F(x, u, u_x)dx$. The authors are able to prove acceptable formulations of the 19th and 20th Hilbert problems.

The general elliptic equation (4) is treated in Chapter 6. Derivative estimates are established under various growth assumptions on the coefficients leading to a discussion of the Dirichlet problem. Strong solutions with bounded first derivatives are shown to be smooth according to the smoothness of the data—a considerably weaker result than may be obtained for the divergence structure case mentioned above.

Many of the results of earlier chapters are extended to systems of equations in Chapters 7 and 8. However, a very specialized form of system is considered which permits an automatic extension of the earlier theory. It is on the theory of general uniformly elliptic systems where much current research interest is focused. In Chapter 9 the authors discuss alternate methods, based on works of Moser, Nirenberg, Morrey and others, for obtaining their estimates of previous chapters. A further approach to the Hölder theory has since been given by the reviewer based on Moser's work on Harnack's inequality. The final chapter treats some other boundary value problems.

The book is in a certain sense complete—the body of theory it presents is close to a final form. It would be of great use for a mathematician already somewhat expert in elliptic partial differential equations. However, for someone seeking an introduction to the theory of quasilinear elliptic equations, the present book would not be the appropriate source. A book of this kind still remains to be written.

N. S. TRUDINGER

Convex sets, by F. A. Valentine. McGraw-Hill Series in Higher Mathematics, McGraw-Hill, New York, 1964. ix+238 pp. $12.00.

In 1934 Bonnesen and Fenchel published a comprehensive survey of the geometry of convex bodies. More than half of it was directed toward quantitative aspects of the theory. By contrast, only a tenth of the present book is devoted to such aspects. This change reflects the developments of the last three decades, which have tended increasingly to emphasize the combinatorial, qualitative, and infinite-dimensional aspects of the theory.

Valentine's book consists of a preface, thirteen chapters (called "parts"), an appendix, a bibliography, and a subject index. Headings of the parts are as follows: I. Basic concepts, II. Hyperplanes and the separation theorem, III. The Minkowski metric, IV. Some char-