# PARTIAL DIFFERENTIAL OPERATORS ON $L^p(E^n)$

BY MARTIN SCHECHTER[1]

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Let $P(\xi)$ be a polynomial in the variables $\xi=(\xi_1, \cdots, \xi_n)$. If we replace $\xi$ by $D=(D_1, \cdots, D_n)$, where $D_j=-i\partial/\partial x_j$, $1 \leqq j \leqq n$, we obtain a constant-coefficient partial differential operator $P(D)$. Acting on the set $C_0^\infty$ of smooth functions with compact supports in Euclidean $n$-dimensional space $E^n$, the operator $P(D)$ is closable in $L^p=L^p(E^n)$ for $1 \leqq p \leqq \infty$. The purpose of this note is to describe some of the spectral properties of its closure $P_{0p}$ in $L^p$.

PROPOSITION 1. *$\sigma(P_{02})$ consists of the closure of the set of values taken on by $P(\xi)$ with $\xi$ real.*

PROPOSITION 2. *A point $\lambda$ is in $\rho(P_{0p})$ if and only if $1/[P(\xi)-\lambda]$ is a multiplier in $L^p$* (cf. [1]).

In applying this proposition we shall let $\mu=(\mu_1, \cdots, \mu_n)$ be a multi-index of nonnegative integers. We set $|\mu|=\mu_1+\cdots+\mu_n$ and

$$P^{(\mu)}(\xi)=\partial^{|\mu|}P(\xi)/\partial\xi_1^{\mu_1}\cdots\partial\xi_n^{\mu_n}.$$

With the aid of a theorem of Littman [1] we obtain

THEOREM 3. *Suppose that $1<p<\infty$, and let $l$ be the smallest integer $>n|1/2-1/p|$. Assume that for $\xi$ real*

$$P^{(\mu)}(\xi)/P(\xi)=O(|\xi|^{-a|\mu|}) \quad \text{as } |\xi|\to\infty, \quad |\mu|\leqq l, \tag{1}$$

$$1/P(\xi)=O(|\xi|^{-b}) \quad \text{as } |\xi|\to\infty, \tag{2}$$

*where $b>(1-a)n|1/2-1/p|$. Then $\lambda\in\rho(P_{0p})$ if and only if $P(\xi)\neq\lambda$ for real $\xi$.*

Let $P(\xi)$ and $Q(\xi)$ be polynomials.

PROPOSITION 4. *A necessary and sufficient condition that $D(P_{02})\subseteq D(Q_{02})$ is that*

$$|Q(\xi)|\leqq C(|P(\xi)|+1), \qquad \xi \text{ real}.$$

PROPOSITION 5. *If $\lambda\in\rho(P_{0p})$, then a necessary and sufficient condition that $D(P_{0p})\subseteq D(Q_{0p})$ is that $Q(\xi)/[P(\xi)-\lambda]$ be a multiplier in $L^p$.*

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THEOREM 6. *Suppose that* $1<p<\infty$ *and that* $P(\xi)$ *and* $Q(\xi)$ *satisfy*

$$P^{(\mu)}(\xi)/P(\xi) = O(|\xi|^{-a|\mu|}) \quad \text{as } |\xi| \to \infty, \quad \text{each } \mu, \tag{3}$$

$$Q(\xi)/P(\xi) = O(|\xi|^{-c}) \quad \text{as } |\xi| \to \infty \tag{4}$$

*for* $\xi$ *real, where* $a \geqq 0$ *and* $c>(1-a)n|1/2-1/p|$. *If* $\rho(P_0)$ *is not empty, then* $D(P_{0p}) \subseteq D(Q_{0p})$.

Let $q(x)$ be a function defined in $E^n$, and let $V$ be the set of those functions $u \in L^p$ such that $qu \in L^p$. Consider multiplication by $q$ as an operator on $L^p$ with domain $V$. This operator is closed; denote it by $q$. For $1 \leqq p < \infty$ and $\alpha$ real, set

$$M_{\alpha,p}(q) = \sup_y \int_{|x-y|<1} |q(x)|^p |x-y|^\alpha dx.$$

THEOREM 7. *Suppose* $P(\xi)$ *satisfies* (2) *and*

$$P^{(\mu)}(\xi)/P(\xi) = O(|\xi|^{-a|\mu|}) \quad \text{as } |\xi| \to \infty, \quad |\mu| \leqq n+1, \tag{5}$$

*with* $b>a+n-an$. *Let* $k_0$ *denote the smallest nonnegative integer satisfying* $ak_0>n-b$. *Assume that* $1 \leqq p<\infty$ *and that* $q(x)$ *is locally in* $L^p$ *and that* $M_{\alpha,p}(q)<\infty$ *for some* $\alpha$ *satisfying* $-n<\alpha<p(n-k_0)-n$. *If* $\rho(P_{0p})$ *is not empty, then* $D(P_{0p}) \subseteq D(q)$. *If in addition*

$$\int_{|x-y|<1} |q(x)|^p dx \to 0 \quad \text{as } |y| \to \infty, \tag{6}$$

*then* $q$ *is* $P_{0p}$*-compact.*

THEOREM 8. *Suppose* $P(\xi)$ *and* $Q(\xi)$ *satisfy* (3) *and* (4) *with* $a \geqq 0$ *and* $c>a+n-an$. *Let* $j_0$ *be the smallest nonnegative integer such that* $aj_0>n-c$. *Assume that* $1 \leqq p<\infty$ *and that* $q(x)$ *is locally in* $L^p$ *and* $M_{\beta,p}(q)<\infty$ *for some* $\beta$ *satisfying* $-n<\beta<p(n-j_0)-n$. *Assume also that* $\rho(P_{0p})$ *is not empty. Then* $D(P_{0p}) \subseteq D(qQ_{0p})$. *If* $q$ *also satisfies* (6), *then the operator* $qQ_{0p}$ *is* $P_{0p}$*-compact.*

## REFERENCE


1. Walter Littman, *Multipliers in* $L^p$ *and interpolation*, Bull. Amer. Math. Soc. **71** (1965), 764–766.


BELFER GRADUATE SCHOOL OF SCIENCE OF YESHIVA UNIVERSITY, NEW YORK, NEW YORK 10033