SMOOTH HOMOTOPY PROJECTIVE SPACES

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Introduction. In [5] we considered certain fixed point free involutions on Brieskorn manifolds as weakly complex bordism elements. In [4] we considered associated examples of smooth normal invariants for real projective spaces, settling the realizability question for dimensions $\not\equiv 1 \bmod 4$ and the desuspendability question for dimensions $4k+1$. The object of this study is the classification of these smooth normal invariants given by the Brieskorn examples. Our results overlap somewhat with Atiyah and Bott [2] as well as Browder [3], but our methods are entirely different and our results rather more refined. Full details of these and related results will appear elsewhere.

1. **Smooth normal invariants.** Following Sullivan [6], we regard a smooth normal invariant of a space X as an element of $[X, G/O]$. Of course, we have $G/O \cong SG/SO$. We need the fibers SG/Spin of $B\text{Spin} \to BSG$ and $SO/\text{Spin} \simeq P^\infty$ of $B\text{Spin} \to BSO$. The spaces SG/SO, SG/Spin, SO/Spin have their Whitney H-space structures under which the sequence

$$SO/\text{Spin} \to SG/\text{Spin} \to SG/SO$$

is a multiplicative fibration.

A map $\mu\colon SG/\text{Spin} \to BO$ is constructed as follows. Let γ_n denote the universal fiber space over BSG_n with fiber S^{n-1}, β_n the pullback to $B\text{Spin}_n$, and α_n the pullback to SG_n/Spin_n; also, let ϵ_n denote the S^{n-1} fibration over a point. Corresponding to the commutative diagram

$$\begin{array}{ccccc} & & B\text{Spin}_n & & \\ & \nearrow & & \searrow & \\ SG_n/\text{Spin}_n & & & & BSG_n \\ & \searrow & & \nearrow & \\ & & \{pt\} & & \end{array}$$

of spaces, there is the commutative diagram

$$\begin{array}{ccccc} & & \beta_n & & \\ & \nearrow & & \searrow & \\ \alpha_n & & & & \gamma_n \\ & \searrow & & \nearrow & \\ & & \epsilon_n & & \end{array}$$

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of induced S^{n-1} fibrations. Passing to Thom spaces we obtain a commutative diagram of spectra

$$\begin{array}{ccccc} & & M\mathrm{Spin} & & \\ & \nearrow & & \searrow & \\ N(SG/\mathrm{Spin}) & & & & MSG \\ & \searrow & & \nearrow & \\ & & \boldsymbol{S} & & \end{array}$$

where $\boldsymbol{S}$ is the sphere spectrum and $N(SG/\mathrm{Spin})$ is the spectrum with $N(SG/\mathrm{Spin})_n = T(\alpha_n) \simeq S^n \wedge (SG_n/\mathrm{Spin}_n)^+$. There is a map of spectra $M\mathrm{Spin} \to \boldsymbol{b}O$, where $\boldsymbol{b}O$ is the Ω-spectrum with $bO_0 = \boldsymbol{Z} \times BO$, defining the kO-orientation of Spin cobordism. Now the composition

$$N(SG/\mathrm{Spin}) \to M\mathrm{Spin} \to \boldsymbol{b}O$$

of maps of ring spectra defines $\mu\colon SG/\mathrm{Spin} \to BO$ in the usual way, since $N(SG/\mathrm{Spin})$ and $\boldsymbol{S} \wedge (SG/\mathrm{Spin})^+$ are equivalent.

(1.1) THEOREM. *The map $\mu\colon SG/\mathrm{Spin} \to BO$ is an H-map from the Whitney structure of SG/Spin to the tensor product structure of BO.*

The following is a central fact in our study.

(1.2) THEOREM. *The composition*

$$P^\infty \simeq SO/\mathrm{Spin} \to SG/\mathrm{Spin} \to BO$$

classifies the canonical line bundle η over P^∞.

Since this map $\eta\colon P^\infty \to BO$ splits the map $w_1\colon BO \to K(Z_2, 1) \simeq P^\infty$, we get the following easily.

(1.3) COROLLARY. *The above maps fit into a commutative diagram*

$$\begin{array}{ccccc} P^\infty & \to & SG/\mathrm{Spin} & \to & G/O \\ \| & & \downarrow \mu & & \downarrow \nu \\ P^\infty & \xrightarrow{\eta} & BO & \to & BSO \end{array}$$

of maps of H-spaces (BO and BSO with the tensor product structure), each row being a multiplicative fibration.

Using Poincaré duality for Spin cobordism theory, we are able to make explicit computations of the kO-orientation $\mu\colon SG/\mathrm{Spin} \to BO$. This amounts to interpreting $N(SG/\mathrm{Spin}) \to M\mathrm{Spin}$ in terms of Spin bordism of Spin manifolds.

For a finite CW complex X, we take a homotopy equivalent compact Spin manifold M^m. An element of $[M^m, SG/\mathrm{Spin}]$ is represented by a spherical fiber bundle over M^m with Spin structural group and an SG (i.e. degree $+1$ fiber homotopy) trivialization. Using transverse regularity on the SG trivialization, we obtain an element $[V^m, \partial V^m; e] \in \Omega_m^{\text{Spin}}(M^m, \partial M^m)$ of degree $+1$ (e is the bundle projection). The Poincaré duality isomorphism sends this to an element of $\Omega^0_{\text{Spin}}(M^m)$ with augmentation $+1$. Now applying the kO-orientation of Spin cobordism, we obtain a virtual line bundle over M^m, i.e. an element of $[M^m, BO]$. This describes the natural map $[X, \mu]$: $[X, SG/\text{Spin}] \to [X, BO]$.

2. **Spin cobordism of projective spaces.** At this point the Brieskorn examples are brought into play. The degree q maps $h: Q_q^{4k-1} \to P^{4k-1}$ of [5, §3] may be taken to represent elements $[Q_q^{4k-1}, h] \in \Omega_{4k-1}^{\text{Spin}}(P^{4k-1})$ in a canonical way so that the following analogue of [5, Theorem 3.4] holds.

(2.1) THEOREM. *The Poincaré duals* $b_q^{4k-1} \in \Omega^0_{\text{Spin}}(P^{4k-1})$ *of the elements* $[Q_q^{4k-1}, h] \in \Omega_{4k-1}^{\text{Spin}}(P^{4k-1})$ *satisfy* $b_q^{4k-1} = q \cdot 1$.

It happens that $h: Q_1^{4k-1} \to P^{4k-1}$ is a diffeomorphism, making it clear what Spin structures to use. Actually, the above holds equally well for Sp and SU cobordism of P^{4k-1}.

To see what happens for other dimensions, just cut the maps $h: Q_q^{4k-1} \to P^{4k-1}$ down to a suitably small smooth regular neighborhood of $P^n \subset P^{4k-1}$, obtaining h: $V_q^{4k-1,n} \to M^{4k-1,n}$, say. Note that the Brieskorn examples Q_q^n of [4] are the transverse regular inverse images in Q_q^{4k-1} of $P^n \subset P^{4k-1}$. The resulting elements $b_q^n \in \Omega^0_{\text{Spin}}(P^n)$ are just the inclusion induced pullbacks of the elements b_q^{4k-1} for $n \leqq 4k-1$.

(2.2) COROLLARY. *In* $\Omega^0_{\text{Spin}}(P^n)$ *we have* $b_q^n = q \cdot 1$.

The double covering $e: S^{4k-1} \to P^{4k-1}$ defines the element $[S^{4k-1}, e] \in \Omega_{4k-1}^{\text{Spin}}(P^{4k-1})$ of degree $+2$. For each n, Poincaré duality and pulling back from P^{4k-1} to P^n produce the elements $s^n \in \Omega^0_{\text{Spin}}(P^n)$ with augmentation $+2$. To emphasize the fact that $s^n \neq 2 \cdot 1$ we point out the following.

(2.3) THEOREM. *Under the kO-orientation morphism*

$$\Omega^*_{\text{Spin}}(P^n) \to kO^*(P^n)$$

$s^n \mapsto 1 + \eta$, *where* η *is the canonical line bundle.*

The following, on the other hand, is straightforward geometry.

(2.4) PROPOSITION. *For any integer j there is a map*

$$h': (V_q^{4k-1,n}, \partial V_q^{4k-1,n}) \to (M^{4k-1,n}, \partial M^{4k-1,n})$$

of degree $q+2j$ *such that the Poincaré dual of* $[V_q^{4k-1,n}, \partial V_q^{4k-1,n}; h']$ *is* $b_q^n + j\cdot s^n \in \Omega_{\mathrm{Spin}}^0(P^n)$.

3. **The Brieskorn normal invariants.** In [4] we pointed out how the Brieskorn examples Q_{2d+1}^n produce smooth normal invariants of P^n. Here we show how these examples lead to elements of $[P^n, SG/\mathrm{Spin}]$ compatible with the machinery of the previous section.

As remarked in [4], each Q_{2d+1}^{4k+1} is homotopy equivalent to P^{4k+1}. Consequently the map

$$h_{2d+1}: (V_{2d+1}^{4k+3,4k+1}, \partial V_{2d+1}^{4k+3,4k+1}) \to (M^{4k+3,4k+1}, \partial M^{4k+3,4k+1})$$

of degree $+1$ provided by (2.4) is a homotopy equivalence of Spin manifolds. Now the construction of [6] produces in our case a "classifying" SG/Spin-bundle over $M^{4k+3,4k+1} \simeq P^{4k+1}$ for h_{2d+1}, i.e. an element $a_{2d+1}^{4k+1} \in [P^{4k+1}, SG/\mathrm{Spin}]$. Restriction to $P^n \subset P^{4k+1}$ defines the element $a_{2d+1}^n \in [P^n, SG/\mathrm{Spin}]$.

Now we follow the description of $[P^n, \mu]$ as given in §1. We find that a_{2d+1}^n leads to the element $b_{2d+1}^n - d\cdot s^n \in \Omega_{\mathrm{Spin}}^0(P^n)$ which by (2.2) and (2.3) is sent (under kO-orientation) to the element $1 + d\cdot\xi \in kO^0(P^n)$, where $\xi = 1 - \eta$. This gives our main result on the Brieskorn examples.

(3.1) THEOREM. *Under kO-orientation we have*

$$[P^n, \mu](a_{2d+1}^n) = 1 + d\cdot\xi$$

as a virtual line bundle over P^n.

Numerous results follow from (3.1), in particular the following which stems from (1.3).

(3.2) THEOREM. *The natural maps*

$$[P^n, \mu]: [P^n, SG/\mathrm{Spin}] \to [P^n, BO],$$
$$[P^n, \nu]: [P^n, G/O] \to [P^n, BSO],$$

are epimorphisms of groups (where BO and BSO have their tensor product structures).

Results of Browder [3] can be applied to show that the epimorphisms in (3.2) are canonically split—in fact, that $[P^n, \nu]$ *classifies* the Brieskorn examples of smooth normal invariants for P^n. Similarly, $[P^n, \mu]$ classifies the Brieskorn elements $a_{2d+1}^n \in [P^n, SG/\mathrm{Spin}]$. Browder's results alone do not give this except for $n \leqq 5$.

4. **Numerical results.** By results of Adams [1], $kO^0(P^n) = KO(P^n)$

is the commutative ring generated by 1 and $\xi = 1 - \eta$ subject to the relations $\xi^2 = 2 \cdot \xi$ and $a_{n+1} \cdot \xi = 0$, where a_k is given by the table:

k	1	2	3	4	5	6	7	$8 \cdots j+8$
a_k	1	2	4	4	8	8	8	$8 \cdots 16a_j$

Now it is clear that the multiplicative group of virtual line bundles $1 + d \cdot \xi \in [P^n, BO]$ is isomorphic to $Z_2 \times Z_{a_{n+1}/2}$. Moreover, the two generators $1 - \xi = \eta$ and $1 - 2 \cdot \xi = 2 \cdot \eta - 1$ of $[P^n, BO]$ generate the Z_2 and $Z_{a_{n+1}/2}$ factors corresponding to $[P^n, P^\infty]$ and $[P^n, BSO]$, respectively—as indicated in the following diagram:

$$\begin{array}{ccccc} [P^n, P^\infty] & \rightleftarrows & [P^n, BO] & \rightleftarrows & [P^n, BSO] \\ \wr\| & & \wr\| & & \wr\| \\ Z_2 & \rightleftarrows & Z_2 \times Z_{a_{n+1}/2} & \rightleftarrows & Z_{a_{n+1}/2} \end{array}$$

(4.1) THEOREM. *There are $a_{n+1}/2$ distinct Brieskorn smooth normal invariants of P^n.*

(4.2) COROLLARY. *There are $a_{4k+2}/2 = 2^{2k}$ smoothly distinct Brieskorn homotopy projective $(4k+1)$-spaces. For $k > 0$, these yield only 4 combinatorially distinct homotopy projective $(4k+1)$-spaces.*

(4.3) COROLLARY. *For $n \not\equiv 1 \bmod 4$ and $n > 5$, there are $a_{n+1}/4$ smoothly distinct homotopy projective n-spaces which yield only 2 combinatorially distinct homotopy projective n-spaces.*

REFERENCES

1. J. F. Adams, *Vector fields on spheres*, Ann. of Math. (2) **75** (1962), 603–632.
2. M. F. Atiyah and R. Bott, *A Lefschetz fixed point formula for elliptic complexes*. II, Ann. of Math. (2) **88** (1968), 451–491.
3. William Browder (to appear).
4. C. H. Giffen, *Desuspendability of free involutions on Brieskorn spheres*, Bull. Amer. Math. Soc. **75** (1969), 426–429.
5. ———, *Weakly complex involutions and cobordism of projective spaces* (to appear).
6. D. P. Sullivan, *Triangulating and smoothing homotopy equivalences and homeomorphisms*, Geometric Topology Seminar Notes (mimeographed), Princeton University, Princeton, N. J., 1967.

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