# THE SECOND HOMOTOPY GROUP OF SPUN 2-SPHERES IN 4-SPACE

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1. **Introduction.** Andrews and Curtis [1] have shown that the second homotopy group of the complementary domain of a locally flat 2-sphere $S^2$ in the 4-sphere $S^4$ may not be trivial. This was shown to be the case if $S^2$ is formed by spinning the trefoil knot. Epstein [3] has shown that if $S^2$ is a spun nontrivial 2-sphere, then $\pi_2(S^4-S^2)$ is a free abelian group of infinite rank. Fox [6] has suggested that it might be more fruitful to consider the second homotopy group with its $\pi_1$-action, and has asked for an algorithm for calculating $\pi_2(S^4-S^2)$ as a $J\pi_1$-module. Sumners [8] has constructed a knotted 2-sphere in $S^4$ for which $\pi_2$ has nontrivial $J\pi_1$-torsion.

The following theorem gives the structure of $\pi_2$ as a $J\pi_1$-module for the case of spun 2-spheres.

THEOREM 2. *If $k(S^2)\subset S^4$ is a 2-sphere formed by spinning an arc $A$ about the sphere $S^2$ and $(x_0, x_1, \cdots, x_n: r_1, r_2, \cdots, r_m)^\phi$ is a presentation of $\pi_1(S^4-k(S^2))$ with $x_0$ the image of the generator of $\pi_1(S^2-A)$ under the inclusion map, then*

$$\left(X_i(1 \leqq i \leqq n): \sum_{i=1}^{n} (\partial r_j/\partial x_i)*X_i = 0 \ (1 \leqq j \leqq m)\right)$$

*is a presentation of $\pi_2(S^4-k(S^2))$ as a $J\pi_1$-module.*

2. **Outline of proof.** Let $S^n$ be the standard $n$-sphere. Let $S^n_\pm$ be the closed domains of $S^n-S^{n-1}$. Let $A$ be an arc in $S^3_+$ which meets $S^2$ only in the end-points of $A$. Now rotate $S^3_+$ about $S^2$. Then $A$ sweeps out a 2-sphere $k(S^2)$ called a spun 2-sphere [2].

THEOREM 1. *If $k(S^2)\subset S^4$ is a spun 2-sphere, then $\pi_2(S^4-k(S^2))$ $\simeq K/[K,K]$, where $K$ is the kernel of the homomorphism $i_*: \pi_1(S^3-k(S^2))$ $\to \pi_1(S^4-k(S^2))$ induced by inclusion and $[K, K]$ is the commutator subgroup of $K$.*

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**Proof in brief.** Let $[f] \in K$, where $f: (S^1, 1) \to (S^3 - kS^2, p)$. Since $[f]$ lies in the kernel of $i_*$, there exists a $g: (S^2, 1) \to (S^4 - kS^2, p)$ such that

(1) $g | S^1 = f$,
(2) $g(S^2_+) \subset S^4_+ - kS^2$,
(3) $g(S^2_-) \subset S^4_- - kS^2$.

Define $\Phi: K \to \pi_2(S^4 - k(S^2))$ as $\Phi[f] = [g]$. It follows from the asphericity of knots [7] that $\pi_2(S^4_\pm - k(S^2)) = 0$, and hence that $\Phi: K \to \pi_2(S^4 - k(S^2))$ is a well-defined homomorphism. It can now be shown that $\Phi$ is onto and has $[K, K]$ as its kernel.

Note that the following sequences are exact:

$$1 \to K \xrightarrow{j_*} \pi_1(S^3 - k(S^2)) \xrightarrow{i_*} \pi_1(S^4 - k(S^2)) \to 1$$
$$1 \to [K, K] \to K \to \pi_2(S^4 - k(S^2)) \to 0.$$

Hence the action of $\pi_1(S^4 - k(S^2))$ on $\pi_2(S^4 - k(S^2))$ is obtained by lifting the elements of $\pi_1(S^4 - k(S^2))$ by $i_*$ to $\pi_1(S^3 - k(S^2))$ and then applying the natural action of $\pi_1(S^3 - k(S^2))$ on its normal subgroup $K$.

Let $(x_0, x_1, x_2, \cdots, x_n : r_1, r_2, \cdots, r_m)^\phi$ be a presentation of $\pi_1(S^4 - k(S^2))$ with $x_0$ representing the image of the generator of $\pi_1(S^2 - k(S^2))$ under the homomorphism $j_*: \pi_1(S^2 - k(S^2)) \to \pi_1(S^4 - k(S^2))$ induced by inclusion. A corresponding presentation of $G = \pi_1(S^3 - k(S^2))$ is $(x_0, x_{\pm 1}, x_{\pm 2}, \cdots, x_{\pm n} : r_{\pm 1}, r_{\pm 2}, \cdots, r_{\pm m})^\phi$, where $r_i(x_0, x_{-1}, \cdots, x_{-n}) = r_{-i}$. Then $K$ is the normal closure of $\{\phi(x_i x_{-i}^{-1})\}$ in $G$.

By means of the Reidemeister-Schreier theorem [5] it can be shown that:

LEMMA 7. $(\{x_\alpha\}_{\alpha \in H} : \{r_\alpha\}_{\alpha \in H}, \{x_{-i\beta}(i \geq 0)\}_{\beta \in H})$ *is a presentation of* $K$, *where* $H = \pi_1(S^4 - k(S^2))$.

Lifting the action of $\pi_1(S^4 - k(S^2))$ on $\pi_2(S^4 - k(S^2))$ up to this presentation, we have Theorem 1.

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# CLASSIFICATION OF KNOTS IN CODIMENSION TWO

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**Introduction.** In this paper we consider smooth knots, i.e., smooth embeddings $\phi: S^n \to S^{n+2}$, $n \geqq 3$. Two knots $\phi$ and $\eta$ are said to be equivalent if there is a diffeomorphism $f: S^{n+2} \to S^{n+2}$ such that $f\phi(S^n) = \eta(S^n)$. The embedding $\phi$ extends to an embedding $\bar{\phi}: S^n \times D^2 \to S^{n+2}$, and any two such extensions are ambient isotopic relative to $S^n \times 0$. Hence if $A = \mathrm{cl}(S^{n+2} - \bar{\phi}(S^n \times D^2))$, the pair $(A, \partial A)$ is determined up to diffeomorphism by the equivalence class of $\phi$. We call $(A, \partial A)$ the complementary pair, or simply the complement, of the knot $\phi$. In this paper we show that if $\pi_1 A$, the fundamental group of the knot, is infinite cyclic, then there is at most one knot inequivalent to $\phi$ with complementary pair $(B, \partial B)$ of the same homotopy type as $(A, \partial A)$. This result is of interest because for any $n \geqq 3$ there are many inequivalent knots $\phi: S^n \to S^{n+2}$ with fundamental group $\boldsymbol{Z}$, see for example [12]. (The result also holds in the P.L. case, provided $\phi$ extends to a P.L.-embedding $\bar{\phi}: S^n \times D^2 \to S^{n+2}$.)

1. **Knots with diffeomorphic complements.** In [4], Gluck showed that homeomorphisms of $S^2 \times S^1$ are isotopic if and only if they are homotopic and used this result to conclude that there are at most two knots $\phi: S^2 \to S^4$ with homeomorphic exteriors. In [1], W. Browder studied the pseudo-isotopy classes of diffeomorphisms (and P.L. equivalences) of $S^1 \times S^n$ for $n \geqq 5$. He showed that two P.L. equivalences are pseudo-isotopic if and only if they are homotopic. For the group $\mathfrak{D}(S^n \times S^1)$ of pseudo-isotopy classes of diffeomorphisms, he obtained the exact sequence