SPACES DETERMINED BY A GROUP OF FUNCTIONS

BY C. L. WIGINTON AND SUSAN SHRADER

Communicated by Orville G. Harrold, May 6, 1968

1. **Introduction.** Let G_F denote the group of all homeomorphisms of the topological space F onto itself, and let $G_{F'}$ be similarly defined for a space F'. If G_F and $G_{F'}$ are topologized under the point open topology, and if there is a function from G_F onto $G_{F'}$ which is a homeomorphism as well as an algebraic isomorphism then Wechsler [1] has determined a sufficient condition for the spaces F and F' to be homeomorphic. Thomas [2] has recently generalized Wechsler's theorem by weakening this condition on the spaces F and F'. It is the purpose here to generalize Wechsler's theorem in a different direction by using a group of functions other than a group of homeomorphisms.

2. **Preliminaries.** Most of our notation can be found in [1] and [2]; for reference we include the following. The space F is *n-homogeneous with respect to a group of functions* G provided for any pair of proper n-tuples $(x_1, \cdots, x_n)$, $(y_1, \cdots, y_n)$, there is a g in G such that $g(x_i)=y_i$, $i=1, \cdots, n$. The space F is *ω-homogeneous with respect to a group of functions* G provided it is n-homogeneous with respect to G for each positive integer n.

Let $G_x=\{f\in G: f(x)=x\}$. Then G_x is a subgroup of G and will be called the *subgroup of the point* x. Furthermore G/G_x will denote the set of left cosets, and cosets will be written as fG_x.

We will use the point open topology on G and will consider G/G_x to have the topology induced by the natural mapping, that is, $\nu_x: G\rightarrow G/G_x$ defined by $\nu_x(h)=hG_x$ is to be continuous so that a set U is open in G/G_x if and only if $\nu_x^{-1}(U)$ is open in G. All spaces are T_2.

Our main theorem is as follows:

THEOREM 1. *Let F be a topological space, and let G denote a group of one-to-one functions from F onto itself with respect to which F is ω-homogeneous, and let F' and G' be similarly defined. Suppose that Φ is a homeomorphism from G onto G' such that Φ is an isomorphism. Then there is a homeomorphism from F onto F'.*

The proof of the main theorem will be accomplished by showing the existence of a sequence of homeomorphisms whose composition will then be the desired homeomorphism between F and F'. We prove first that G/G_x is homeomorphic to F. We then show that Φ induces a homeomorphism from G/G_x onto $G'/\Phi(G_x)$. It is next shown that the subgroup $\Phi(G_x)$ is, in fact, the subgroup of a point y in F', i.e. $\Phi(G_x) = G'_y$. We have then that $G'/\Phi(G_x) = G'/G'_y$. Finally, an application of the first result gives that G'/G'_y is homeomorphic to F'. Thus, letting $\simeq$ denote the relation of homeomorphism, we have

$$F \simeq G/G_x \simeq G'/\Phi(G_x) = G'/G'_y \simeq F'$$

and therefore F is homeomorphic to F'.

3. **The sequence of homeomorphisms.** A subbasis element in G is denoted by $W(y, U) = \{f \in G: f(y) \in U\}$ where y is a point of F and U is an open set in F. A basis element is denoted by $W(\bar{y}, \overline{U})$ where $\bar{y} = (y_1, \cdots, y_n)$ is a proper n-tuple and $\overline{U} = (U_1, \cdots, U_n)$. The symbol $W(y)$ is used to denote $\{z \in F: g(y) = z, g \in W\}$.

LEMMA 1. *The function $\mu_x: G \to F$ defined by $\mu_x(h) = h(x)$ is continuous and open.*

PROOF. This is shown in [1] for a group G of homeomorphisms. It is easily seen that the continuity and openness of elements of G are not required.

THEOREM 2. *The function $\theta_x: G/G_x \to F$ defined by $\theta_x(hG_x) = h(x)$ is a homeomorphism.*

PROOF. The comment in the proof of Lemma 1 applies here also.

Suppose F' is a Hausdorff topological space which is ω-homogeneous with respect to a group G' of one-to-one functions from F' onto itself. Suppose further that there is a homeomorphism Φ from G to G' which is an isomorphism. Let $H = G_x$ and $H' = \Phi(G_x)$.

LEMMA 2. *Using the notation of the previous paragraph, Φ induces a homeomorphism between G/H and G'/H' defined as $\phi = \nu'\Phi\nu_x^{-1}$, where ν' is the natural map from G' onto G'/H'.*

PROOF. Trivial.

Our next task is to show that H' is the subgroup of a point $y \in F'$. For this we will need several lemmas.

LEMMA 3. *If $f \in G - H$, then HfH is dense in G.*

PROOF. The proof is contained in Theorem 3.1 of [2] where again the continuity and openness of elements of G are not required.

LEMMA 4. *H' is a proper closed subgroup of G'.*

PROOF. Trivial.

Throughout the remainder $W(\bar{x}, \overline{U})$ will denote a nonempty basic open set in $G' - H'$ such that $\bar{x} \in F'^n$, $\overline{U} \subset F'^n$ and n is the smallest integer with this property. If $\bar{w}=(w_1, \cdots, w_n)$ denote $(w_1, \cdots, w_{p-1}, w_{p+1}, \cdots, w_n)$ by $\bar{w}_p$.

LEMMA 5. *If $n>1$ then $H'(\bar{x}_k)$ is dense in F'^{n-1} and therefore infinite in each component.*

PROOF. If there is an open set $O\subset F'^{n-1}$ which does not intersect $H'(\bar{x}_k)$, then $W(\bar{x}_k, O)$ contradicts the minimality of n.

LEMMA 6. *If $n>1$ there exist two independent points in $H'(\bar{x})$.*

PROOF. The proof is as in Lemma 3.14 of [1].

LEMMA 7. *Suppose there is an n-tuple $\bar{z}$ in $H'(\bar{x})$ independent of $\bar{x}$. Then $\bar{x}$ lies in infinitely many distinct sets of the form $g(H'(\bar{x}))$ where g is in G'.*

PROOF. This proof is obtained in the same manner as the similar result in [2, Lemma 2.4].

THEOREM 3. *For some y in F', $H'=G'_y$.*

PROOF. Having established the preceding Lemmas, the proof given in [2, Theorem 3.1] applies.

4. **Proof of Theorem 1.** From Theorem 2, we have that F and G/G_x are homeomorphic and so also are F' and G'/G'_y. That G/G_x and $G'/\Phi(G_x)$ are homeomorphic is a consequence of Lemma 2. Theorem 3 establishes the equality of $G'/\Phi(G_x)$ and G'/G_y. Thus

$$F \simeq G/G_x \simeq G'/\Phi(G_x) = G'/G'_y \simeq F'$$

and therefore F and F' are homeomorphic.

REFERENCES

1. M. T. Wechsler, *Homeomorphism groups of certain topological spaces*, Ann. of Math. 62 (1955), 360–373.

2. E. S. Thomas, Jr., *Spaces determined by their homeomorphism groups*, Trans. Amer. Math. Soc. 126 (1967), 244–250.

UNIVERSITY OF HOUSTON