ON THE GROWTH OF $f(g)$

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It is well known [1] that if f and g are transcendental and entire, then

$$T(r, f(g))/T(r, f) \to \infty \qquad \text{as } r \to \infty. \tag{1}$$

It is reasonable to conjecture that (1) remains valid when f is assumed to be meromorphic instead of entire. (Here and in the sequel it is assumed that the reader is familiar with the Nevanlinna functions $T(r, f)$, $N(r, f)$, $m(r, f)$, etc.)

Using some results of Edrei and Fuchs [2] one can easily verify for any given $\epsilon > 0$ that

$$T(r, f(g)) > \frac{1-\epsilon}{3} T(r, f) \tag{2}$$

for sufficiently large r and for certain families of functions $\{f\}$ and $\{g\}$. For example (2) holds when g is transcendental of finite order and f is transcendental and meromorphic with at least two zeros.

One can also derive other weak results of this type out of Nevanlinna's second fundamental theorem. It seems, however, that anything stronger must be derived from something somewhat more precise than Nevanlinna's theorem.

In this note we show how an extension of the second fundamental theorem can be used to prove (1) for a large class of meromorphic functions g.

Nevanlinna's theorem can be made more precise [3] as follows:

THEOREM 1. *Suppose $f(z)$ is a nonconstant meromorphic function with $f(0) \neq 0, \infty$. For any sequence $a_1, a_2, \cdots$, $|a_i| \leqq |a_{i+1}|$, with $a_i \neq f(0)$, let $\delta(k)$ = minimum of the distances between the first k points of the sequence. Then for every $k \geqq 2$,*

$$\begin{aligned}(k-1)T(r,f) \leqq\ & N(r,f) + \sum_{i=1}^{k} N\left(r, \frac{1}{f-a_i}\right) \\ & - \left\{2N(r,f) + N\left(r, \frac{1}{f'}\right) - N(r,f')\right\} \\ & + c'k \log r + r(k-1) \log (1/\delta(k)) \\ & + O(\log rT(r, f))\end{aligned} \tag{3}$$

for all r in $S(k)$ outside a set E_0 of r of finite measure. E_0 and $O(\log (rT(r, f))$ do not depend on k. Here $S(k) = \{r; r > |a_k| \}$ and c' is a constant.

Note that the term involving $k \log k$ which we find in Nevanlinna's remainder [4] does not appear in (3).

Using Theorem 1 one can easily prove

THEOREM 2. *Let $f(z)$ be a transcendental meromorphic function such that $f(0) \neq 0, \infty$. If a discrete set*

$$S = \{a_1, a_2, \cdots \}, \qquad a_i \neq f(0),$$

with three or more elements satisfies for some t,

$$\delta(n(r, S)) \geqq r^{-t}$$

for a set, E, of r of infinite measure, then for every $\epsilon > 0$,

$$n(r, f^{-1}(S)) \geqq (1 - \epsilon)(n(r, S) - 2)T(r, f)/\log r$$

for all r in E outside a set of finite measure. Here $n(r, S)$ denotes the number of elements of S in $|z| \leqq r$.

We are now ready to prove

THEOREM 3. *Let f be meromorphic, transcendental and such that for three distinct numbers A_i, $i = 1, 2, 3$*

$$\delta(n(r, f^{-1}(A_i))) \geqq r^{-t}$$

for some t and for all r outside a set of finite measure. If g is transcendental and such that $g(0) \notin f^{-1}(A_i)$; $i = 1, 2, 3$, then (1) holds as r approaches infinity outside a set of finite measure.

PROOF. We may assume without any loss of generality that $g(0) \neq 0$. It follows from Theorem 2 that for any given real number K and any function g and set S satisfying the hypotheses of Theorem 2

$$n(r, g^{-1}(S)) > Kn(r, S) \tag{4}$$

for all sufficiently large r outside the exceptional set. The measure of the exceptional set does not depend on K.

We use (4) to find the relationship between $N(r, 1/(f(g) - A_i))$ and $N(r, 1/(f - A_i))$; $i = 1, 2, 3$, that we need for the proof.

Let $S_i = f^{-1}(A_i)$. Then $f(g(z)) = A_i$ if and only if $z \in g^{-1}(S_i)$. Thus it follows from (4) that

$$n\left(r, \frac{1}{f(g) - A_i}\right) > Kn\left(r, \frac{1}{f - A_i}\right). \tag{5}$$

Thus $N(r, 1/(f(g)-A_i))>KN(r, 1/(f-A_i))$ outside a set of r of finite measure. By applying Nevanlinna's second fundamental theorem, we obtain for any $\epsilon>0$

$$T(r, f(g)) > \frac{(1-\epsilon)}{3} KT(r, f) \tag{6}$$

outside a set of r of finite measure and our theorem follows.

A similar but somewhat more elaborate argument can be used to prove the more general:

THEOREM 4. *If f and g are as in Theorem 3 and p is any polynomial, then*

$$T(r, f(g))/T(r, f(p)) \to \infty$$

outside a set of r of finite measure.

p in Theorem 4 can probably be replaced by any entire function whose growth compared to that of g is sufficiently small.

As an immediate consequence of Theorem 4 we have

COROLLARY 1. *If f and g are as in Theorem 4 and*

$$f(g) = f(p), \tag{7}$$

where p is a polynomial, then g is a polynomial of the same degree as p.

COROLLARY 2. *If f is meromorphic, transcendental and of finite order and if p and g are as in Theorems 3 and 4, then the conclusions of Theorems 3 and 4 remain valid.*

PROOF. We need only note that when f is of finite order, it satisfies the hypotheses of both theorems.

PROOF. It follows from Theorem 4 that g is a polynomial. The fact that it has the same degree is proved in [**5**].

The solutions g of (7) are known when f is entire (see [**5**]). It is not even known, however, whether the above corollary is true for arbitrary mermomorphic f.

Added in Proof. REMARK. Subsequent to the completion of this paper J. Clunie completed a rather extensive study on this and related problems. His results should appear soon in Mcintyres Memorial Volume (University of Ohio).

REFERENCES

1. W. K. Hayman, *Meromorphic functions*, Oxford Mathematical Monographs, Clarendon Press, Oxford 1964, Chapter 2.

2. A. Edrei and W. H. Fuchs, *On the zeros of $f(g(z))$ where f and g are entire functions*, J. Analyse Math. **12** (1964), 243–255.

3. F. Gross, *On the distribution of values of meromorphic functions*, Trans. Amer. Math. Soc. **131** (1968), 199–214.

4. R. Nevanlinna, *Le théorème de Picard-Borel et la théorie des fonctions méromorphes*, Gauthier-Villars, Paris, 1929.

5. N. I. Baker and F. Gross, *On factorizing entire functions*, J. London Math. Soc. (1968), 69–76.

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Anthony J. Schaeffer. *Boundeness of solutions to linear differential equations*, pp. 508–511.

Page 508: The third formula should read:

$$Q(t)A(t) + A^*(t)Q(t) + \dot{Q}(t) = 0.$$

Page 509: Equation (3) should read:

$$Q(t)A(t) + A^*(t)Q(t) + \dot{Q}(t) = 0.$$

Page 509: The first formula in the proof of Theorem 1 should read:

$$\left(\frac{d}{dt}\right) \langle Q(t)x(t), x(t) \rangle$$

$$= \langle \dot{Q}(t)x(t, x(t) + \cdots$$

$$= \langle [\dot{Q}(t) + \cdots .$$

Page 510: The first line should read:

$$\dot{Q}_\tau(t) = \dot{X}^*(\tau, t)X(\tau, t) + X^*(\tau, t)X(\tau, t).$$

Page 510: The second equation in the second remark should read:

$$y(t) = [P(t)A(t)P^{-1}(t) + \dot{P}(t)P^{-1}(t)]y(t) = B(t)y(t).$$