A UNIFORM GENERALIZED SCHOENFLIES THEOREM

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The generalized Schoenflies theorem of M. Brown [2], [3] can be restated in the following way: If S^{n-1} is the equator of S^n, then any locally flat embedding $f: S^{n-1} \to S^n$ can be extended to a homeomorphism $F: S^n \to S^n$.

The purpose of this paper is to show that, if $n \geqq 5$, the extension F can be constructed in a controlled manner; in particular, if $f: S^{n-1} \to S^n$ is close to the inclusion embedding, then $F: S^n \to S^n$ can be chosen to be close to the identity homeomorphism. Consequently if, $f, g: S^{n-1} \to S^n$ are locally flat embeddings, $n \geqq 5$, and f is close to g, then there is a homeomorphism $H: S^n \to S^n$ which is close to the identity such that $Hf = g$.

Let S^{n-1} denote the unit sphere in E^n, B^n the unit ball, and O the origin. If x, y belong to $E^n - O$, let $\theta(x, y)$ denote the angle in radians between the line segments Ox and Oy, measured such that $0 \leqq \theta(x, y) \leqq \pi$. The distance between x and y under the Euclidean metric will be denoted by dist (x, y). If A is a subset of $E^n - O$, the angular diameter of A, written θ diam A, is defined to be $\sup_{x,y \in A} \theta(x, y)$. This is significant whenever A lies in a half-space.

Now suppose S is a locally flatly embedded $(n-1)$-sphere in E^n which approximates the standard sphere S^{n-1}. Suppose $\phi: S^{n-1} \times [0, 1] \to \mathrm{Cl}(\mathrm{Ext}\ S)$ is a collar on S in Cl(Ext S). If the collar is small, then the θ-diameter of each fiber $\phi(x \times [0, 1])$ is also small. The object of Lemma 2 is to push the collar outward, leaving S fixed, so that its two boundary components are separated by a round sphere with center at O, and so that the θ-diameter of each fiber remains small. The precise statement is as follows.

LEMMA 2. *If $f: S^{n-1} \to E^n$, $n \geqq 5$, is a locally flat embedding such that for all $x \in S^{n-1}$, $\theta(x, f(x)) < \epsilon$, where $\epsilon < \pi/7$, then there is an embedding $F: S^{n-1} \times [0, 1] \to \mathrm{Cl}(\mathrm{Ext}\, f(S^{n-1}))$ such that:*

(1) *$F(x, 0) = f(x)$,*

(2) *$F(S^{n-1} \times 0)$ and $F(S^{n-1} \times 1)$ are separated by some round sphere with center at O,*

(3) *For all $x \in S^{n-1}$, $t \in [0, 1]$, $\theta(x, F(x, t)) < 13n\epsilon/2 + 15\epsilon$.*

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The proof of Lemma 2 requires four auxiliary lemmas. We begin with a collar ϕ on $S=f(S^{n-1})$, and let $U=\phi(S^{n-1}\times(0, 1))$. Then U is an open subset of Ext S. We further assume $\theta(x, \phi(x, t))<\epsilon$ for all x and t. Lemma A states that any complex in Ext S can be pulled into U (in the sense of [1]) by a homotopy in Ext S whose orbits have θ-diameter at most 9ϵ. Lemma B states that any complex in Ext S can be disentangled from S, i.e., pulled into the exterior of some round sphere Σ outside S, by a homotopy in Ext S whose orbits have θ-diameter less than 4ϵ. The condition that $\theta(x, f(x))<\epsilon$ for all $x\in S^{n-1}$ insures that the "folds" in S are small, and hence any point of Ext S may be moved into U or outside Σ along a path of small θ-diameter. The condition $\epsilon<\pi/7$ is a purely artificial one which makes the proofs work.

Lemmas 1A and 1B are radial engulfing lemmas. The engulfings proceed along the orbits of the homotopies guaranteed by Lemmas A and B. The proofs of these lemmas are almost identical to the proof of Engulfing Theorem A of [1], and their functions are comparable to those of Lemmas 1 and 2 of [5].

Finally, the proof of Lemma 2 is accomplished in the manner of Lemma 9.1 of [6].

If S_1 and S_2 are disjoint locally flat $(n-1)$-spheres in E^n, $S_1\subset \operatorname{Int} S_2$, and if there is a stable homeomorphism $h: E^n\to E^n$ such that $h(S_1)=S_2$, then S_1 and S_2 cobound an annulus (Theorem 10.3 of [4]). We next strengthen a special case of this theorem.

Let $\bar{S}$ be a sphere concentric with S^{n-1}. Let $\bar{x}$ denote the point of $\bar{S}$ which is coradial with $x\in S^{n-1}$. Introduce the following notation: if $y\in E^n-O$ and L is a real number such that $\|y\|+L>0$, then $y+L$ denotes the unique point of E^n which is coradial with y and has norm $\|y\|+L$.

If $f: S^{n-1}\to E^n$ and $\bar{f}: \bar{S}\to E^n$ are embeddings, we say that f and $\bar{f}$ are *parallel* if there is a real number L such that for all $x\in S^{n-1}$, $f(x)=\bar{f}(\bar{x})+L$.

Clearly any two disjoint parallel spheres are stably equivalent, hence cobound an annulus. Lemma 3 states that this annulus can be coordinatized so that the θ-diameters of the fibers are directly proportional to the θ-deviation of f itself.

LEMMA 3. *Let $\bar{S}$ be a sphere concentric with S^{n-1}, of radius less than* 1. *Let A be the annulus between $\bar{S}$ and S^{n-1}. Let $0<\epsilon<\pi/7$, and let $f: S^{n-1}\to E^n$, $n\geq 5$, be a locally flat embedding such that $\theta(x, f(x))<\epsilon$ for all $x\in S^{n-1}$. Suppose $\bar{f}: \bar{S}\to \operatorname{Int} f(S^{n-1})$ is an embedding which is parallel to f. Then there is an embedding $F: A\to E^n$ such that:*

(1) $F|S^{n-1}=f$,
(2) $F|\bar{S}=\bar{f}$,
(3) $\theta(y, F(y))<(39/2)n\epsilon+45\epsilon$, *for all* $y\in A$.

To prove Lemma 3, apply Lemma 2 to obtain an annulus in $\mathrm{Cl}(\mathrm{Ext} f(S^{n-1}))$ which satisfies the conclusion of Lemma 2. Call this annulus R_1, and denote the annulus between $f(S^{n-1})$ and $\bar{f}(\bar{S})$ by R_2. Using the fact that Int R_1 contains a round sphere with center at O, push R_1 onto $R_1\cup R_2$ by a radial homeomorphism of E^n. This does not alter the θ-diameters of the fibers of R_1. Next, map $R_1\cup R_2$ homeomorphically onto R_2 by utilizing the annular structure on R_1. This at worst triples the θ-diameters of fibers. The result of these maps gives an annular structure on R_2 satisfying Lemma 3.

The main theorems.

THEOREM 1. *If* $n\geqq 5$, *and* $f: S^{n-1}\to E^n$ *is a locally flat embedding such that* $\theta(f(x), x)<\epsilon$ *and* dist $(f(x), x)<\epsilon$ *for all* $x\in S^{n-1}$, *then* f *can be extended to an embedding* $F: B^n\to E^n$ *such that* dist $(F(x), x) <39n\epsilon/2+48\epsilon$.

COROLLARY 1. *For each* $\eta>0$, *there is a* $\delta>0$ *such that each locally flat* δ*-embedding of* S^{n-1} *into* E^n, $n\geqq 5$, *can be extended to an* η*-embedding of* B^n *into* E^n.

The proof of Theorem 1 is outlined as follows. Partition B^n into annuli A_i of thickness 2ϵ together with a small ball B_* in the center. Partition $\mathrm{Cl}(\mathrm{Int} f(S^{n-1}))$ into annular regions R_i, together with a small cell C about the origin, in such a way that each boundary sphere of each R_i is parallel to $f(S^{n-1})$ and the parallel distance between any two consecutive spheres (i.e., the constant L of the definition of parallel embeddings) is 2ϵ. Obtain a 1-1 correspondence between the A_i and the R_i by omitting the innermost A_i or R_i, if necessary. Use Lemma 3 to map the outermost annulus A_0 homeomorphically onto the outermost region R_0. (We assume $\epsilon<\pi/7$, for if not, Theorem 1 is certainly true.) Then it is possible to map each A_i onto the corresponding R_i by copying the map $f|A_0$. This procedure is well defined on $A_i\cap A_{i+1}$, because of the parallel condition. Finally, map B_* homeomorphically onto C in any fashion, extending the map $F|\dot{B}_*$.

For points $y\in A_i$, $|\,\|y\|-\|F(y)\|\,|<3\epsilon$ and $\theta(y, F(y))<(39/2)n\epsilon+45\epsilon$. Since $\|y\|\leqq 1$, $\mathrm{dist}(y, F(y))<(39/2)n\epsilon+48\epsilon$. For points $y\in B_*$ no control is necessary because $B_*\cup C$ has diameter less than 7ϵ.

Now consider S^{n-1} to be the equator of S^n. Theorems 2 and 3 follow from Corollary 1.

THEOREM 2. *Let* $n \geqq 5$, $\eta > 0$. *There is a* $\delta > 0$ *such that any locally flat* δ*-embedding* $f: S^{n-1} \to S^n$ *can be extended to a* η*-homeomorphism* $F: S^n \to S^n$.

THEOREM 3. *Let* $n \geqq 5$, $\eta \geqq 0$. *Let* $g: S^{n-1} \to S^n$ *be any locally flat embedding. There exists a* $\delta > 0$ *such that if* $f: S^{n-1} \to S^n$ *is any locally flat embedding satisfying* $\operatorname{dist}(f(x), g(x)) < \delta$, *then there is an* η*-homeomorphism* $H: S^n \to S^n$ *such that* $Hf = g$.

These results, together with those of Connell [5] and Bing [1], can be used to show that the problem of approximating homeomorphisms of S^n, $n \geqq 5$, by p.w.l. ones is equivalent to approximating locally flat embeddings of $(n-1)$-spheres by p.w.l. ones.

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