FREE PIECEWISE LINEAR INVOLUTIONS ON SPHERES

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If T is a piecewise linear fixed-point free involution on S^n, the orbit space $Q^n = S^n/T$ is a PL-manifold homotopy equivalent to $P_n(\boldsymbol{R}) = P^n$ [2]; the affirmative solution to the Poincaré conjecture implies that conversely for $n \neq 3, 4$ the double covering manifold of any such Q^n can be identified with S^n. Write I_n for the set of (oriented if n is even) PL-homeomorphism classes of manifolds Q^n homotopy equivalent to P^n. We will compute I_n for $n \neq 3, 4$.

Let Q^n be as above. We define a normal invariant $\eta(Q)$. Take a homotopy equivalence $h: P^n \to Q^n$ (orientation-preserving if n is odd): this is unique up to homotopy. Approximate $h \times 0$ by a PL-embedding $P^n \times Q^n \to \boldsymbol{R}^N (N > n)$; let ν^N be the normal bundle of the embedding, which exists if N is large enough [5], and $F: \nu^N \to \epsilon^N$ the fibre homotopy trivialisation induced by the homotopy equivalence [7], [10, 3.5]. Then (ν, F) induces a homotopy class $\eta(Q)$ of maps $P^n \to G/\mathrm{PL}$, which depends only on the PL-homeomorphism class of Q. We have thus defined $\eta: I_n \to [P^n, G/\mathrm{PL}]$: our description follows Sullivan [8], the main idea goes back to Novikov [6].

We next compute $[P^n, G/\mathrm{PL}]$. The homotopy groups of G/PL are known to be $\boldsymbol{Z}$ (in dimensions $4i$), $\boldsymbol{Z}_2$ (in dimensions $4i+2$), and 0 (in odd dimensions). Further, Sullivan [8] has shown that if finite groups of odd order are ignored, the only nonzero k-invariant is the first (which is δSq^2). We choose fundamental classes $x_{2i} \in H^{2i}(G/\mathrm{PL}; \boldsymbol{Z}_2)$ $(i \neq 2)$, $\alpha \in H^1(P^n; \boldsymbol{Z}_2)$. Because of the k-invariant, $[P^4, G/\mathrm{PL}] \cong \boldsymbol{Z}_4$: let y be an isomorphism. Further, denote by r the restriction $[P^{n+1}, G/\mathrm{PL}] \to [P^n, G/\mathrm{PL}]$. Then we have

LEMMA 1. *Let $i \geqq 0$. Then we have bijections*

$$[P^{2i+5}, G/\mathrm{PL}] \overset{r}{\cong} [P^{2i+4}, G/\mathrm{PL}] \overset{X}{\cong} \boldsymbol{Z}_4 \oplus \sum_{1 \leq j \leq i} \boldsymbol{Z}_2,$$

where the components of X are $[y] = yr^{2i}$ and $[x_{2j+4}]$ with

$$[x_{2j+4}](f) = f^*(x_{2j+4})\alpha^{2i-2j}[P_{2i+4}].$$

Moreover, $[x_2]$ is the mod 2 *reduction of $[y]$.*

We compute the image and 'kernel' of η by surgery: in fact we have abelian groups $L_n(\boldsymbol{Z}_2^+)$ and $L_n(\boldsymbol{Z}_2^-)$ (the second referring to the nonorientable case) such that for $n \geq 3$ the image of $\eta: I_{2n} \to [P^{2n}, G/\mathrm{PL}]$ is the kernel of a map ϑ to $L_{2n}(\mathbf{Z}_2^-)$, and the fibres of η are the orbits of an action of $L_{2n+1}(\mathbf{Z}_2^-)$ on I_{2n}; similarly for I_{2n-1}. These results will appear in [11]; also the following sharpening of a result of [9].

LEMMA 2. *We have* $c: L_{2i}(\mathbf{Z}_2^-) \cong \mathbf{Z}_2$, $L_{2i+1}(\mathbf{Z}_2^-) = 0$,

$$(\sigma/8, \tilde{\sigma}/8): L_{4i}(\mathbf{Z}_2^+) \cong \mathbf{Z} + \mathbf{Z},\ L_{4i+1}(\mathbf{Z}_2^+) = 0,\ c: L_{4i+2}(\mathbf{Z}_2^+) \cong \mathbf{Z}_2,$$

and d: $L_{4i+3}(\mathbf{Z}_2^+) \cong \mathbf{Z}_2$, *where* c *denotes Kervaire-Arf invariant,* σ *signature and* $\tilde{\sigma}$ *signature of double covers.*

The next stage in the argument is the explicit computation of c. Results of Sullivan [8] and Browder, which are too long to summarize here, show that c (see [9, 4.7]) can be defined as an invariant of bordism class in $\mathfrak{N}_*^{\mathrm{PL}}(G/\mathrm{PL})$, and satisfies a product formula. Moreover, if the x_{4i+2} are suitably chosen, there exists a stable cohomology operation

$$a = 1 + Sq^2 + Sq^2Sq^2$$

such that if $k = a(\sum_{i \geq 0} x_{4i+2})$, and $f: M^{2r} \to G/\mathrm{PL}$, then

$$c(f) = w(M) \cdot f^*(k)[M].$$

Applying this to $f: P^n \to G/\mathrm{PL}$, we find that if $n = 4i-2$ or $4i$, $i \geq 2$, then for suitable $a_{nj} \in \mathbf{Z}_2$,

$$\text{(c)} \qquad c(f) = [x_{4i-2}] + \sum_{j<i} a_{nj}[x_{4j-2}].$$

Finally, we will need the suspension. Let $n \geq 5$, and let $Q^n \simeq P^n$. Then the double cover, $\tilde{Q}^n$, of Q^n is PL-homeomorphic to S^n. Hence if we attach (by the projection map) the cone on $\tilde{Q}^n$ to Q^n, we have a closed $(n+1)$-manifold homotopy equivalent to P^{n+1}. We call this the suspension ΣQ; the definition stems from Browder and Livesay [1].

LEMMA 3. *The following diagram is commutative:*

$$\begin{array}{ccccc}
I_{4i-2} & \xrightarrow{\eta} & [P^{4i-2}, G/\mathrm{PL}] & & \\
\downarrow \Sigma & & r \uparrow \cong & \searrow c_{4i-2}\vartheta & \\
I_{4i-1} & \xrightarrow{\eta} & [P^{4i-1}, G/\mathrm{PL}] & \xrightarrow{d\vartheta} & \mathbf{Z}_2 \\
\downarrow \Sigma & & r \uparrow & \nearrow c_{4i}\vartheta & \\
I_{4i} & \xrightarrow{\eta} & [P^{4i}, G/\mathrm{PL}] & &
\end{array}$$

The result follows from (c) and diagram chasing. A more direct proof has been found by Santiago López de Medrano [12].

THEOREM. *We have the following bijections, for* $i \geqq 1$,

$$I_{4i+2} \overset{\Sigma^{-1}}{\cong} I_{4i+1} \overset{\eta}{\cong} [P^{4i+1}, G/\mathrm{PL}] \overset{X}{\cong} Z_4 \oplus \sum_{3 \leq j \leq 2i} Z_2,$$

$$(r, \tau): I_{4i+3} \cong I_{4i+2} \oplus Z,$$

$$(r^2, [x_{4i+4}]): I_{4i+4} \cong I_{4i+2} \oplus Z_2.$$

The suspension $I_{4i+3} \to I_{4i+4}$ is the sum of the identity and the epimorphism $Z \to Z_2$.

Our computation of surgery obstructions has already determined $\operatorname{Im} \eta$ in all cases. Further, η is injective on I_n for n even (since $L_{n+1}(Z_2^-) = 0$), and also if $n = 4i+1$, using the fact that $L_{4i+2}(1) \to L_{4i+2}(Z_2^+)$ is bijective and a known argument in the simply connected case. The same argument shows for $n = 4i+3$ that the image of $L_{4i+4}(1) \to L_{4i+4}(Z_2^+)$ operates trivially on I_{4i+3}, so we need only consider the quotient group, which is detected by the invariant $2\sigma - \tilde{\sigma}$.

We now define a map τ: $I_{4i-1} \to Z$. Any homotopy projective space Q^{4i-1} determines naturally a bordism element $x \in \Omega_{4i-1}^{\mathrm{PL}}(P_\infty(R))$. This group is finite, so for some integer N, NQ bounds W^{4i} say; let $\tilde{W}$ be the double cover of W induced from $P_\infty(R)$. Since each component of ∂W is a rational homology sphere, we can define a signature $\sigma(W)$, also $\sigma(\tilde{W})$. Since (by the Hirzebruch formula) the signature of a double covering of a *closed* oriented PL manifold is twice that of the manifold, $\tau(Q) = \{2\sigma(W) - \sigma(\tilde{W})\}/8N$ depends only on Q. Then τ induces a bijection of each fibre of η onto Z.

REMARK 1. Each of our invariants, reduced mod 2, is a topological invariant of Q_n. Except for τ, this follows from the proofs (due, respectively, to Casson (unpublished) and Sullivan [8]) of the Hauptvermutung for 1-connected manifolds with $H_3(M; Z)$ free of 2-torsion. For τ(mod 2) we use the suspension (which is defined topologically) and the last part of Theorem 4.

REMARK 2. For 4-manifolds, arguments analogous to the above show that there are at most 2 h-cobordism classes of manifolds $Q_4 \simeq P_4$; if there are 2, they are distinguished by $y\eta$ (whose image will then be $Z_2 \subset Z_4$). Even if the nontrivial class exists, it is not clear whether the double cover of any manifold in it will be S^4 itself.

For 3-manifolds, it is known that if $\tilde{Q}_3 = S^3$, then $Q_3 = P_3$ (Livesay [3]). In lower dimensions, all becomes trivial.

REMARK 3. Orientation-reversal of P_{2i+1} comes from reflection in P_{2i}, and hence induces the identity on $[P_{2i}, G/\mathrm{PL}]\overset{r^{-1}}{\cong}[P_{2i+1}, G/\mathrm{PL}]$. However, it clearly alters the sign of τ. Thus all our manifolds admit orientation-reversing PL-homeomorphisms except those of dimension $4k+3$ with $\tau\neq 0$.

REMARK 4. Browder and Livesay have developed [1] an obstruction theory for the kernel and cokernel of Σ. In their exact sequences we have

$$0\to I_{4k}\xrightarrow{\Sigma} I_{4k+1}\xrightarrow{f_1} Z_2 \qquad f_1 = c\theta r\eta,$$

$$Z_2\xrightarrow{f_2} I_{4k+1}\xrightarrow{\Sigma} I_{4k+2}\to 0 \qquad f_2 \text{ is zero } (\Sigma \text{ is injective}),$$

$$0\to I_{4k+2}\xrightarrow{\Sigma} I_{4k+3}\xrightarrow{f_3} Z \qquad f_3 = \tau.$$

$$Z\xrightarrow{f_4} I_{4k+3}\xrightarrow{\Sigma} I_{4k+4}\to 0$$

$f_4(n)Q=Q'$, where Q and Q' have the same normal invariant and $\tau(Q')=\tau(Q)+2n$.

I am indebted to D. Sullivan and to F. Hirzebruch for supplying proofs of the result about f_3. Surjectivity of f_3 was also shown by S. López de Medrano [4].

REMARK 5. $P_{2n+1}(R)$ is fibred over $P_n(C)$ with fibre S^1. The projec- induces an epimorphism

$$[P_n(C), G/\mathrm{PL}]\to [P^{2n+1}, G/\mathrm{PL}].$$

Given any (PL-) manifold M^{2n} homotopy equivalent to $P_n(C)$, we can take the induced fibration, and obtain an element of I_{2n+1}. Moreover, it has been shown by Sullivan (for the Z_2 case) and proved by Montgomery and Yang, and also by Hirzebruch (for the Z-case) that the Browder-Livesay obstruction above to desuspending the element of I_{2n+1} induces the obstruction to desuspending analogously M^{2n}. Hence conversely, an element Q_{2n+1} of I_{2n+1} comes from the total space of such a fibration if and only if

$(n=2)$. $Q_5=P_5$.

$(n=3)$. The normal invariant of Q_7 is given by a certain function of $\tau(Q_7)$.

$(n=2k,\ k>1)$. A mod 2 obstruction $z_{4k}=[x_{4k}]+$lower terms vanishes.

$(n=2k+1,\ k>1)$. z_{4k} equals the mod 2 reduction of τ.

We cannot compute z_{4k} precisely since we have given no exact definition of the x_{4k}.

REMARK 6. The problems of smooth free involutions on S^n and on homotopy spheres remain unsolved. It is easy to obtain partial results —for example, S. López de Medrano has observed [12] that smooth homotopy projective spaces are determined modulo $\Theta_n(\partial\pi)$ by the smooth normal invariant and by τ (this follows easily from Lemma 2 above). We will not attempt to summarize all that is known, but refer to [2] for one of several known constructions which have yet to be fitted in with the viewpoint of this paper.

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