ALMOST EVERYWHERE CONVERGENCE OF POISSON INTEGRALS ON GENERALIZED HALF-PLANES

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Communicated by A. P. Calderón, December 6, 1967

1. **Introduction.** A classical theorem of Fatou states that if f is an L^p function on the line (circle), $p \geqq 1$, and if the harmonic function F on the upper half-plane (disk) is the Poisson integral of f, then $F(z) \to f(x)$ as $z \to x$ nontangentially for a.e. x on the line (circle).

Generalizations in several directions have recently been found, e.g. [1], [2], [4], [6]. Our result, stated precisely below, is Fatou's theorem for generalized upper half-planes holomorphically equivalent to bounded symmetric domains and functions of type L^p, $p > 1$, or locally of type $L \log {}^+L$. Details will appear elsewhere.

In §2, we sketch the setting and state our result explicitly. The proof is case-by-case, and includes the case of the exceptional domains; §3 is devoted to a sketch of the proof in a typical case.

2. **The theorem.** Let D be a generalized upper half-plane, i.e.

$$D = \{(z, w) \in V_1 \times V_2 \colon \operatorname{Im} z - \Phi(w, w) \in \Omega\},$$

where V_1 is a complex vector space with a given real form, V_2 is a complex vector space, $\Omega \subset \operatorname{Re} V_1$ is an open cone, and $\Phi \colon V_2 \times V_2 \to V_1$ is hermitian symmetric bilinear with respect to $\operatorname{Re} V_1$ such that $\Phi(w, w) \in \bar{\Omega}$. When Ω is a domain of positivity and Φ satisfies certain symmetry and homogeneity properties, D is holomorphically equivalent to a bounded symmetric domain [5]. The distinguished boundary of D is

$$B = \{(z, w) \colon \operatorname{Im} z - \Phi(w, w) = 0\}.$$

We identify B with $\operatorname{Re} V_1 \times V_2$ by associating to $(x + i\Phi(w, w), w)$ the pair (x, w). There is a nilpotent group $\mathfrak{N}$ of automorphisms of D which acts transitively on B and is also equal to $\operatorname{Re} V_1 \times V_2$ as a set. Haar measure on $\mathfrak{N}$ is the induced Euclidean measure.

The Poisson kernel, $P(u, \zeta)$, is defined on $B \times D$, and the Poisson integral of a function f on B is

$$F(\zeta) = \int_B f(u) P(u, \zeta) du.$$

[1] Partially supported by the U. S. Air Force Office of Scientific Research; Harmonic Analysis Contract at Princeton University.

For details of the above, see [3].

For $u\in B$, $t\in\Omega$ we write $u_t=u+(it, 0)$. Also, let I be the base point in Ω.

THEOREM 1. *Let D be a generalized upper half-plane holomorphically equivalent to a bounded symmetric domain. Suppose that $f\in L^p(B)$, $p>1$, or that $f\in L\log^+ L$ locally and is bounded off a bounded set. Then*

$$\lim_{r\to 0} F(u_{rI}) \to f(u) \text{ for almost every } u \in B.$$

There are more general types of convergence. We say that $t\in\Omega$ approaches 0 *restrictedly* if t is constrained to lie in a proper subcone of Ω. And we say that $u_t\to u_0=g_0\cdot 0$ *admissibly* if u_t stays in some

$$\Gamma_\alpha(u_0) = \{g_0 g\cdot(it, 0): g = (a, c), \max\{|a|, |c|^2\} < \alpha|t|\}.$$

THEOREM 2. *Under the hypothesis of Theorem 1, $F(u_t)\to f(u_0)$ for a.e. $u_0\in B$ as $u_t\to u_0$ admissibly and restrictedly.*

3. **The proof.** The proof for domains D which are tube domains, i.e. for which $V_2=0$, is contained in [6]. The remaining domains, with the exception of one of dimension 16, fall into two large classes, type I and type IIIb. We indicate the proof of Theorems 1 and 2 for domains of type I. The complete proof requires only slight modification.

There is a domain of type I for each pair of integers n, m, $n>0$, $m\geqq 0$. As a bounded domain, it is realized as the space of complex $n\times(n+m)$ matrices ζ satisfying $\zeta\zeta^*<I$. In the realization we consider, V_1 is the complexification of the real vector space of hermitian symmetric $n\times n$ matrices, V_2 is the space of complex $n\times m$ matrices, Ω is the cone of positive definite matrices, and $\Phi(w, w_1)=ww_1^*$. Thus

$$D = D_{n,m} = \{(x + iy, w): y - ww^* > 0\}.$$

The Poisson integral $F((g\cdot 0)_t)$ is shown to be dominated by a sum of maximal functions $f^*_{(j)(k)I}(g)$, $g\in\mathfrak{N}$. We define these.

Let (j) and (k) be, respectively, n-tuples and m-tuples of nonnegative integers and $t>0$.

$$R^t_{(j)} = \{r = (r_1, \cdots, r_n) \in E_n: |r_i| \leqq 2^{j_i}t\},$$
$$S^t_{(k)} = \{s \in E_m: |s_i| \leqq 2^{k_i}t\}.$$

Every $x\in\operatorname{Re} V_1$ can be written in the form $x=k^{-1}d(r)k$, $k\in U(n)$, $r\in E_n$, where $d(r)$ is the diagonal $n\times n$ matrix whose entries are the r_i. And every $w\in V_2$ can be written $w=u\tilde{d}(s)v$, $u\in U(n)$, $v\in U(m)$,

$s \in E_m$, where $\tilde{d}(s)$ is the $n \times m$ diagonal-form matrix whose entries are the s_i. We set

$$H^t_{(j)(k)} = \{(x, w) = (k^{-1}d(r)k, u\tilde{d}(s)v) : r \in R^t_{(j)}, s \in S^{t^{1/2}}_{(k)}\}.$$

(If $m \geqq n$, the rectangles S lie in E_n, but this makes no difference.)

There is defined for each (j) and (k) a sequence of neighborhoods

$$U(n) = N_{(j)(k)0} \supset N_{(j)(k)1} \supset \cdots \supset \{I\},$$

and we define

$$E^t_{(j)(k)l} = \{(x, w) = (k^{-1}d(r)k, u\tilde{d}(s)v) \in H^t_{(j)(k)} : ku \in N_{(j)(k)l}\},$$

$$f^*_{(j)(k)l}(g) = \sup_{t>0} | E^t_{(j)(k)l}|^{-1} \int_{E(j)(k)l} f(gh)dh.$$

We abuse notation now by writing t for both a positive number and the matrix tI.

LEMMA 1.

$$\sup_{t>0} F((g\cdot 0)_t) \leqq A \sum_{(j)(k)} 2^{-[(1/2)|j|+|k|]} \sum_{l=1}^{L} f^*_{(j)(k)l}(g),$$

where L depends only on m and n.

PROOF. It is enough to prove the inequality when $g = 0$. We notice that

$$(1) \qquad P((x, w), (it, 0)) = P_t(x, w) = \left\{\frac{\det t}{|\det(x + i[ww^* + t])|^2}\right\}^{n+m}.$$

The method of proof is to compare the size of $P_t(u), u \in E^t_{(j)(k)l-1} - E^t_{(j-1)(k-1)l}$, with the size of $|E^t_{(j)(k)l}|$. (The $t^{1/2}$ factor in the definition of $E^t_{(j)(k)l}$ is due to the ww^* term in (1).) The necessity for considering the neighborhoods $N_0 \supset N_1 \supset \cdots$ may be seen by considering a special case.

In particular, if $x = k^{-1}d(2^j, 0, \cdots, 0)k$, $w = u\tilde{d}(2^{j/2}, 0, \cdots, 0)v$, then $|\det(x + i[ww^* + I])|^2$ depends on ku, ranging from $2^{2j} + (2^j+1)^2$ when $ku = I$ to $(2^{2j}+1)(2^j+1)^2$. The proof that the number of N_j that need be considered is finite involves an induction, and is complicated.

The proof of Theorem 1 is now routine once one establishes

LEMMA 2. $\|f^*_{(j)(k)l}\|_p \leqq A_p\|f\|_p$, *where* A_p *is independent of* (j), (k) *and* l.

PROOF. It is not hard to show that

$$(2)\qquad f^*_{(j)(k)l}(g) \leqq A \frac{\int_{U(n)}\int_{U(n)}\int_{U(m)} \bar{f}_{(j)(k)}(g;\, k,\, u,\, v)\chi(ku)\,dv\,du\,dk}{\iiint \chi(ku)\,dv\,du\,dk},$$

where χ is the characteristic function of $N_{(j)(k)l}$ and

$$\bar{f}_{(j)(k)}(g;\, k,\, u,\, v) = \sup_{t>0} \left| R^t_{(j)} \right|^{-1} \left| S^{t^{1/2}}_{(k)} \right|^{-1} \int_{R^t_{(j)}} \int_{S^{t^{1/2}}_{(k)}} f(g\cdot(k^{-1}d(r)k,\, u\bar{d}(s)v))\,ds\,dr.$$

The function $\bar{f}(\cdot\,;\, k,\, u,\, v)$ may be thought of as giving maximal averages over $m+n$-dimensional rectangles "pointed" in the direction determined by u, v and k. Now the subgroup

$$\mathfrak{H}_{k,u,v} = \{h = (k^{-1}d(r)k,\, u\bar{d}(s)v) : r \in E_n,\, s \in E_m\}$$

of $\mathfrak{N}$ is isomorphic to $E_n \times E_m$, and so $\bar{f}_{(j)(k)}$ restricted to the coset $g\cdot\mathfrak{H}_{k,u,v}$ is an ordinary maximal function. Thus

$$(3)\qquad \int_{\mathfrak{H}_{k,u,v}} \left| \bar{f}_{(j)(k)}(gh;\, k,\, u,\, v) \right|^p dh \leqq B_p \int_{\mathfrak{H}_{k,u,v}} \left| f(gh) \right|^p dh.$$

Integrating over $\mathfrak{N}/\mathfrak{H}$ on both sides of (3), one has

$$\int_{\mathfrak{N}} \left| \bar{f}(g;\, k,\, u,\, v) \right|^p dg \leqq B_p \int_{\mathfrak{N}} \left| f(g) \right|^p dg.$$

This, together with (2), proves the lemma.

The proof that $\|f^*_{(j)(k)l}\|_1 \leqq A_1 \|f\|_{L \log^+ L}$ depends on the analogous result for ordinary maximal functions. To prove Theorem 1 in the case $f \in L^1$ would involve establishing a weak-type inequality,

$$\left| \{g : |f^*_{(j)(k)l}(g)| > s\} \right| < A_0 s^{-1} \|f\|_1.$$

Since the weak-type inequality for rectangular maximal functions cannot be "rotated" the way norm inequalities can, further analysis is necessary. This analysis has been performed by E. M. Stein (see [7]) and the author; and will appear.

We conclude by noting that Theorem 2 is a consequence of Theorem 1 and the following result, which is a slight extension of the corresponding result in the tube domain case.

LEMMA 3. *Suppose that* $u_t \to u_0$ *restrictedly and admissibly. Let* $\bar{t} > 0$ *be the smallest eigenvalue of* t. *Then, for any* $u' \in B$,

$$P(u', u_t) \leqq AP(u', (u_0)_{\bar{t}I}).$$

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