BERRY-ESSEEN BOUNDS FOR THE MULTI-DIMENSIONAL CENTRAL LIMIT THEOREM[1]

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1. Introduction. Let $\{X_n\}$ be a sequence of independent and identically distributed random variables each with mean zero, variance unity, and finite absolute third moment β_3. Let F_n denote the distribution function of $(X_1+\cdots+X_n)/(n)^{1/2}$. Berry [2] and Esseen [4] have proved that

$$(1)\quad \sup_{x\in R_1} \left| F_n(x) - \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{x} e^{-y^2/2}dy \right| \leqq c\beta_3/n^{1/2}, \qquad n = 1, 2, \cdots,$$

where c is a universal constant. Consider now a sequence $\{X(n) = (X_1^{(n)}, \cdots, X_k^{(n)})\}$ of independent and identically distributed random vectors in R_k each with mean vector $(0, \cdots, 0)$ and covariance matrix I, the $k\times k$ identity matrix. If P_n denotes the probability distribution of $(X^{(1)}+\cdots+X^{(n)})/n^{1/2}$ and Φ is the standard k-dimensional normal distribution, then it is well known that P_n converges weakly to Φ as $n\to\infty$. Bergström [1] has extended (1) to this case, assuming finiteness of absolute third moments of the components of $X^{(1)}$. Since weak convergence of a sequence Q_n of probability measures to Φ means that $Q_n(B)\to\Phi(B)$ for every Borel set B satisfying $\Phi(\partial B)=0$, ∂B being the boundary of B, it seems natural to seek bounds of $|P_n(B)-\Phi(B)|$ for such sets B (called *Φ-continuity sets*). Let $\mathfrak{a}$ be a class of Borel sets such that, whatever be the sequence Q_n converging weakly to Φ, $Q_n(B)\to\Phi(B)$ as $n\to\infty$ uniformly for all $B\in\mathfrak{a}$. Such a class is called a *Φ-uniformity* class. By a theorem of Billingsley and Topsoe [3], a class $\mathfrak{a}$ is a Φ-uniformity class if and only if $\sup\{\Phi(\partial B)^\epsilon; B\in\mathfrak{a}\}\downarrow 0$ as $\epsilon\downarrow 0$, where $(\partial B)^\epsilon$ is the ϵ-neighborhood of ∂B. This leads one naturally to consider the class $\mathfrak{a}_1(d, \epsilon_0)$ of all Borel sets B for which $\Phi(\partial B)^\epsilon \leqq d\epsilon$ for $0<\epsilon<\epsilon_0$, d and ϵ_0 being any two given positive constants. One may also consider the class $\mathfrak{a}_1^*(d, \epsilon_0)$, which is the largest translation-invariant subclass of $\mathfrak{a}_1(d, \epsilon_0)$; this means that $B\in\mathfrak{a}_1^*(d, \epsilon_0)$ if and only if all translates of B belong to $\mathfrak{a}_1(d, \epsilon_0)$.

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2. **Results.** We shall write $\beta_s = \sum_{i=1}^{k} E|X_i^{(1)}|^s$ for $s>0$. Also c's will denote constants. For example, $c_1(k, \delta)$, $c_2(k)$, and c_4 will stand for a constant depending only on k and δ, a constant depending on k alone, and a universal constant, respectively.

THEOREM 1. *Suppose $\beta_{3+\delta} < \infty$ for some $\delta > 0$. Then, for all n,*

$$\sup | P_n(B) - \Phi(B) | \leqq n^{-1/2}\{c_1(k, \delta)\beta_{3+\delta}^{3(1+\delta)/(3+\delta)} + [c_2(k)d + c_3(k)/\epsilon_0]\beta_{3+\delta}^{3/(3+\delta)}\},$$

where the supremum extends over all B in $\mathcal{A}_1^*(d, \epsilon_0)$.

We shall state two applications of Theorem 1.

EXAMPLE 1. Let $\mathcal{C}$ be the class of all measurable convex sets in R_k. It follows from certain results of Ranga Rao [5] that $\mathcal{C} \subset \mathcal{A}_1^*(d(k), \epsilon_0)$ for every $\epsilon_0 > 0$, $d(k)$ being an appropriate constant depending on k. Hence

$$\sup_{C \in \mathcal{C}} | P_n(C) - \Phi(C) | \leqq n^{-1/2}\{c_1(k, \delta)\beta_{3+\delta}^{3(1+\delta)/(3+\delta)} + c_2(k)d(k)\beta_{3+\delta}^{3/(3+\delta)}\}$$

for all n. This is an improvement on a result of Ranga Rao [6].

EXAMPLE 2. Let $\mathcal{F}(l)$ be the class of all measurable sets in R_2 each of whose boundaries is contained in a rectifiable curve of length not exceeding l. It may be shown (cf. [3]) that $\mathcal{F}(l) \subset \mathcal{A}_1^*(4\pi l + 8\pi, 1)$. Hence Theorem 1 applies. In fact, in this case it suffices to assume that $\beta_3 < \infty$, so that we have

$$\sup_{F \in \mathcal{F}(l)} | P_n(F) - \Phi(F) | \leqq n^{-1/2}\{(c_4 l^2 + c_5 l + c_6)\beta_3\}, \quad n = 1, 2, \cdots.$$

THEOREM 2. *Suppose $\beta_{3+\delta} < \infty$ for some $\delta > 0$. Then, for all n,*

$$\sup | P_n(B) - \Phi(B) | \leqq n^{-1/2}\{c_7(k, \delta)\beta_{3+\delta}^{6/(3+\delta)} + c_8(k)[d + 1/\epsilon_0]\beta_{3+\delta}^{3/(3+\delta)} \log(n+1)\},$$

where the supremum extends over all B in $\mathcal{A}_1(d, \epsilon_0)$.

The methods used in proving Theorems 1 and 2 enable one to obtain bounds for general Φ-uniformity classes, and, in particular, for any Φ-continuity set.

An asymptotic expansion holds for the class $\mathcal{A}_1(d, \epsilon_0)$ under the assumption that

$$\limsup_{|t| \to \infty} | f(t) | < 1,$$

where f is the characteristic function of $X^{(1)}$. If $\beta_s < \infty$ for some integer $s \geqq 3$, then $P_n(B)$ may be estimated by this expansion with an error $O(n^{-(s-2)/2} \cdot [\log n]^{k/2})$ uniformly for all $B \in \mathfrak{A}_1(d, \epsilon_0)$.

EXTENSIONS. Theorems 1 and 2 may be extended to the following cases: (1) $\{X^{(n)}\}$ is not identically distributed, but $\sup_n \sum_{i=1}^k E|X_i^{(n)}|^{3+\delta} < \infty$ for some $\delta > 0$; (2) $\{X^{(n)}\}$ has a common nonsingular covariance matrix perhaps different from I.

In proving Theorem 1 we look at the convolution $(P_n - \Phi)*\Gamma_n$, where Γ_n is a probability measure having a characteristic function which vanishes everywhere outside a sphere, and Γ_n converges weakly to the probability measure degenerate at $(0, \cdots, 0)$. Theorem 2 is obtained by sharpening a technique of Esseen [4] and Ranga Rao [5].

The details and proofs of these results, which are part of the author's doctoral dissertation, submitted to the University of Chicago, will appear elsewhere.

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