RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited. Manuscripts more than eight typewritten double spaced pages long will not be considered as acceptable.

A MAXIMAL PROBLEM IN HARMONIC ANALYSIS. III

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1. **Introduction.** Let G be a compact group. Let Σ denote the set of all equivalence classes of continuous irreducible unitary representations of G. For each $\sigma\in\Sigma$, let $U^{(\sigma)}$ be a fixed member of σ. Let H_σ be the [finite-dimensional] Hilbert space on which $U^{(\sigma)}$ acts, and let d_σ denote the dimension of H_σ. Let $\mathfrak{E}(\Sigma)$ denote the product space $P_{\sigma\in\Sigma}\mathfrak{B}(H_\sigma)$. For $f\in\mathfrak{L}_1(G)$, the Fourier transform $\hat{f}$ is the element of $\mathfrak{E}(\Sigma)$ such that

$$\langle \hat{f}(\sigma)\xi,\eta\rangle = \int_G \overline{\langle U_x^{(\sigma)}\xi,\eta\rangle} f(x)dx$$

for all $\xi,\eta\in H_\sigma$ and $\sigma\in\Sigma$.

For an operator A on a finite-dimensional Hilbert space, $|A|$ denotes the unique positive-definite square root of $AA^\sim$ [$\sim$ denotes adjoint]. If $a_1,\cdots,a_n$ denote the eigenvalues of $|A|$, then $\|A\|_{\phi_p}$ denotes $(\sum_1^n a_k^p)^{1/p}$ for $1\leqq p<\infty$ and $\|A\|_{\phi_\infty}$ denotes $\max\{a_k: 1\leqq k\leqq n\}$ $=$ operator norm of A. Let E be an element in $\mathfrak{E}(\Sigma)$. Following R. A. Kunze [4], we define

$$\|E\|_p = \left(\sum_{\sigma\in\Sigma} d_\sigma \|E_\sigma\|_{\phi_p}^p\right)^{1/p}$$

for $1\leqq p<\infty$, and $\|E\|_\infty=\sup\{\|E_\sigma\|_{\phi_\infty}:\sigma\in\Sigma\}$. Finally, we define $\mathfrak{E}_p(\Sigma)=\{E\in\mathfrak{E}(\Sigma):\|E\|_p<\infty\}$ for $1\leqq p\leqq\infty$.

Kunze [4] has proved the following Hausdorff-Young theorems [in considerably greater generality]:

A. *If* $f\in\mathfrak{L}_p(G)$, $1\leqq p\leqq 2$, *and* $1/p+1/p'=1$, *then* $\hat{f}\in\mathfrak{E}_{p'}(\Sigma)$ *and*
(a) $\|\hat{f}\|_{p'}\leqq\|f\|_p$.

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B. *If* $g \in \mathfrak{L}_{p'}(G)$, $1 \leqq p \leqq 2$, *and* $1/p+1/p'=1$, *then*
(b) $\|g\|_{p'} \leqq \|\hat{g}\|_{p}$.

The maximal problem is the problem of determining when identity holds in (a) and (b). For groups that are locally compact and Abelian and $1<p<\infty$, this problem is solved in [1]. For compact groups and $1<p<\infty$, I. I. Hirschman, Jr., [3] solved the maximal problem using different p-norms in $\mathfrak{E}(\Sigma)$; we relate his results to ours in §3.

2. **The main theorems.** A function f on G is a *subcharacter* if there is an open subgroup G_0 of G and a continuous 1-dimensional character χ of G_0 such that $f(x)=\chi(x)$ for $x \in G_0$ and $f(x)=0$ for $x \notin G_0$.

THEOREM 1. *If f is a multiple of a translate of a subcharacter of G, then $\|\hat{f}\|_{p'}=\|f\|_p$ for all p, $1 \leqq p \leqq \infty$.*

THEOREM 2. *Suppose that f belongs to $\mathfrak{L}_p(G)$ where $1<p<\infty$ and $p \neq 2$, and suppose that $\|\hat{f}\|_{p'}=\|f\|_p$. [Such functions are said to be maximal functions.] Then f is a multiple of a translate of a subcharacter.*

3. **Remarks.** For $1<p<\infty$, Hirschman [3] used the following norms:

$$|E|_p = \left(\sum_{\sigma \in \Sigma} d_\sigma^{2-p/2} \|E_\sigma\|_{\phi_2}^p \right)^{1/p}.$$

For $1<p \leqq 2$, we have $|E|_{p'} \leqq \|E\|_{p'}$ and $\|E\|_p \leqq |E|_p$. For $1<p<\infty$ and f in the center $\mathfrak{L}_1^z(G)$ of $\mathfrak{L}_1(G)$, the equality $|\hat{f}|_p = \|\hat{f}\|_p$ obtains. Therefore Hirschman's maximal functions are necessarily maximal with Kunze's norm. Hirschman's maximal functions are just the multiples of translates of subcharacters in $\mathfrak{L}_1^z(G)$.

The proof of Theorem 1 is not difficult. The proof of Theorem 2 is long and rather technical. In broad outline, the proof follows that of Hirschman [3], but certain new difficulties arise. Many of these arise from the fact that our maximal subcharacters need not be in $\mathfrak{L}_1^z(G)$, and so $\hat{f}(\sigma)$ can be complicated: for $f \in \mathfrak{L}_1^z(G)$, each $\hat{f}(\sigma)$ is a multiple of the identity operator. Because of this, some tedious lemmas about operators on finite-dimensional spaces are required.

Another interesting difference is the following. In both treatments, the theorem for $p'>2$ is reduced to the theorem for $p<2$ by means of the following duality property: *f is maximal in $\mathfrak{L}_p(G)$, $1<p<2$, if and only if $F=|f|^{p-1}\operatorname{sgn} f$ is maximal in $\mathfrak{L}_{p'}(G)$.* Hirschman is able to give an explicit form for $\hat{F}$ in terms of $\hat{f}$. All we can prove is that for each $\sigma \in \Sigma$, there exist unitary operators V_σ and W_σ on H_σ such that $\hat{F}(\sigma)=V_\sigma|\hat{f}(\sigma)|^{1/(p-1)}W_\sigma$.

The details of the proofs will be provided in the forthcoming monograph [2].

REFERENCES

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