subgroup K not in the center such that G/K is a subgroup of the symmetric group on 7 elements S_7.

c. The proof of Theorem 1 is similar to [2] and will appear elsewhere.

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GRADED ALGEBRAS, ANTI-INVOLUTIONS, SIMPLE GROUPS AND SYMMETRIC SPACES

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Communicated by David A. Buchsbaum, August 25, 1967

We will outline two different generalisations of the Brauer group of a field (of characteristic $\neq 2$), and a third which combines them. By specialising to real and complex fields, we obtain algebra which describes the (classical) real Lie groups and symmetric spaces. The theory below can be generalised to arbitrary commutative rings; details will appear elsewhere.

We first recall the theory of the classical Brauer group of a field k[2]. Consider simple finite-dimensional k-algebras with centre k. Any such algebra can be written as a matrix ring $M_n(D)$, where D is a division ring with centre k. Write $M_n(D) \sim D$: this induces an equivalence relation. The class of algebras is closed under tensor product. Multiplication is compatible with the equivalence relation, hence induces a product on the set of equivalence classes. For this product, the class of k acts as unit, and taking the opposite algebra gives an inverse. We have an abelian group, $B(k)$.

We next define the graded Brauer group [3], $GB(k)$, of a field k of characteristic $\neq 2$. Here consider ($\mathbf{Z}_2$-) graded k-algebras $A = A_0 \oplus A_1$ of finite dimension, with no proper graded ideals, and such that the intersection of A_0 with the centre of A is k: we call these graded central simple algebras over k. We induce an equivalence relation by $A \sim A \otimes M_n(k)$, where $M_n(k)$ is graded by regarding it as the endomorphism ring of a graded vector space $k^n = V_0 + V_1$, for some grading

[1] To obtain an abstract simple group it suffices in most cases to take the quotient of the commutator subgroup of G by its centre: see [5].

of k^n. The class of algebras is closed under graded tensor product, which respects the equivalence relation, and again the equivalence classes from a group, (the inverse comes from the graded opposite algebra, defined by taking the opposite and changing the sign of the product $A_1 \times A_1 \to A_0$), which we call $GB(k)$. This has a filtration $GB(k) \supset GB^+(k) \supset B(k) \supset \{1\}$; the first two quotients respectively are isomorphic to $\mathbf{Z}_2$ and to $k^*/(k^*)^2$.

For our second generalisation, we replace the field k by a field-with-involution (K, j): here, j is an involutory automorphism of K. Two 'degenerate' cases are included: j may be the identity, or K may be a 'double field'—i.e. the sum $k+k$ of two isomorphic fields, interchanged by j. We denote the fixed field of K under j by k. Assuming characteristic $\neq 2$, there is a good Galois theory for involuted fields (K, j): corresponding to an algebraically closed field $\bar{k}$ we must take $\bar{k}+\bar{k}$, with the 'swap' s; then if G, g are the Galois groups of $\bar{k}$ over K, k respectively, the Galois group of $(\bar{k}+\bar{k}, s)$ over (K, j) lies between $G \times \{1\}$ (the group of $(K+K, s)$) and $g \times \{1, s\}$ (the group of $(k, 1)$). Now consider pairs (A, J) where A is central simple over K, and J an anti-involution (i.e. involutory antiautomorphism) of A inducing j on the centre K of A. Given two such, we can form the tensor product *over* K to obtain a third. We obtain an equivalence relation by making $(A, J) \sim (A, J) \oplus (M_n(K), j_0)$, where j_0 takes a matrix to its adjoint with respect to some nonsingular (j-)hermitian form on K. As above, equivalence classes form an abelian group (the opposite algebra still yields an inverse), and we denote this by $B(K, j)$. In the case of double fields, the algebras occurring are the 'doubles' of central simple algebras over k, so that $B(k+k, s) \cong B(k)$.

Finally we define the graded Brauer group $GB(K, j)$ by considering a graded central simple K-algebra A with an anti-involution J inducing j on the centre. The equivalence relation and tensor product theory extend in an obvious manner to this case. We have $GB(k+k, s) \cong GB(k)$, and in general there is a filtration $GB(K, j) \supset GB^+(K, j) \supset B(K, j) \supset \{1\}$, where the first quotient is $\mathbf{Z}_2$, and the second is $k^*/(k^*)^2$ if $j \neq$ identity, but is $k^*/(k^*)^2 \oplus \mathbf{Z}_2$ if $j=1$. As in the other case, we can also determine the extensions explicitly. We note also in this case an involution I obtained from the grading by setting $I(x_0+x_1)=x_0-x_1$. Since J is an anti-involution of a graded algebra, it commutes with I. Then $J'=IJ$ is another anti-involution. We call (A, J') the *dual* of (A, J). Duality gives an involution of $GB(K, j)$, but induces the identity on $GB(k+k, s)$.

Each of our Brauer groups is clearly a functor on the category of fields (or involuted fields) and embeddings.

We illustrate these definitions by computing the most important examples. As is well known, $B(\mathbf{C})=\{1\}$, $B(\mathbf{R})\cong \mathbf{Z}_2$, and by [**3**], $GB(\mathbf{C})\cong \mathbf{Z}_2$ and $GB(\mathbf{R})\cong \mathbf{Z}_8$. For the others, we must first classify the involuted fields in question. The Galois group over $(\mathbf{R}, 1)$ of its closure $(\mathbf{C}+\mathbf{C}, s)$ is a four group $\{1, s, c, sc\}$, where c gives complex conjugation on each summand. This has 3 proper subgroups, and the fixed field of s is $(\mathbf{C}, 1)$, of c is $(\mathbf{R}+\mathbf{R}, s)$, and of sc is $(\mathbf{C}, c)$. We have $B(\mathbf{R}, 1)\cong \mathbf{Z}_2+\mathbf{Z}_2$, $B(\mathbf{C}, 1)\cong \mathbf{Z}_2$, $B(\mathbf{C}, c)=\{1\}$, $B(\mathbf{R}+\mathbf{R}, s)\cong \mathbf{Z}_2$, $B(\mathbf{C}+\mathbf{C}, s)=\{1\}$, $GB(\mathbf{R}, 1)\cong \mathbf{Z}_8+\mathbf{Z}_4$, $GB(\mathbf{C}, 1)\cong \mathbf{Z}_8$, $GB(\mathbf{C}, c)\cong \mathbf{Z}_2+\mathbf{Z}_2$, $GB(\mathbf{R}+\mathbf{R}, s)\cong \mathbf{Z}_8$, $GB(\mathbf{C}+\mathbf{C}, s)\cong \mathbf{Z}_2$. All maps between these groups induced by field extensions are surjective; $b: B(\mathbf{R}, 1)\to B(\mathbf{C}, 1)\oplus B(\mathbf{R}+\mathbf{R}, s)$ is an isomorphism, and $GB(\mathbf{R}, 1)\to GB(\mathbf{C}, 1)\oplus GB(\mathbf{R}+\mathbf{R}, s)\to GB(\mathbf{C}+\mathbf{C}, s)$ is short exact.

Given a central simple algebra (A, J) over (K, j), we define groups $G=\{x\in A: J(x)\cdot x=1\}$ and $SG=\{x\in G: Nrd(x)=1\}$.

These are reductive and with some low dimensional exceptions, simple modulo their centres (as algebraic groups);[1] the centre of SG is finite. For the fields above, we obtain simple real Lie groups; conversely, all nonexceptional real simple Lie groups correspond (up to isogeny) to some such algebra. Thus for $B(\mathbf{C}+\mathbf{C}, s)$ we have the groups $\mathrm{GL}(\mathbf{C})$; for $B(\mathbf{C}, 1)$ we have $O(\mathbf{C})$ and $\mathrm{Sp}(\mathbf{C})$, for $B(\mathbf{C}, c)$ we have U, and for $B(\mathbf{R}+\mathbf{R}, s)$ we have $\mathrm{GL}(\mathbf{R})$ and $\mathrm{GL}(\mathbf{H})$. We name the four types corresponding to $B(\mathbf{R}, 1)$, using the isomorphism b above, as $O(\mathbf{R})$, $\mathrm{Sp}(\mathbf{R})$, $O(\mathbf{H})$ and $\mathrm{Sp}(\mathbf{H})$. This differs slightly from standard notation, which studiously avoids mentioning $\mathbf{H}$. To name actual groups, rather than types, we need a suffix, except for U, $O(\mathbf{R})$, and $\mathrm{Sp}(\mathbf{H})$, where two suffices are necessary. It is amusing to note a formula for the rank. Exclude odd orthogonal groups over $\mathbf{R}$ and $\mathbf{C}$. Then $4(rk_{\mathbf{R}}G)^2=\dim_{\mathbf{R}}A\ \dim_{\mathbf{R}}K$.

The above shows that any simple algebraic group over $\mathbf{C}$, of classical type, comes from a simple algebra A; the same works for any algebraically (or separably) closed field, also for semisimple groups and algebras. Using 'descent' it can now be shown that any simple algebraic group of classical type (note that we regard triality, etc., as 'nonclassical') comes, up to isogeny, from a simple algebra (A, J). The 'field of definition,' however, is not K but k. For this, see [**4**].

We now interpret elements of $GB(K, J)$. There, the involution, I, gives an involution of the group G above, whose fixed subgroup $H=G\cap A_0$ is the group defined by the algebra $(A_0, J|A_0)$. If $k=\mathbf{R}$ or $\mathbf{C}$, the space G/H is a symmetric space, in general pseudo-Riemannian. Conversely, all such spaces with G a classical group appear on our list: we have 54 types, corresponding to the elements of the graded

Brauer groups of $(\boldsymbol{R}, 1)$, $(\boldsymbol{C}, 1)$, $(\boldsymbol{C}, c)$, $(\boldsymbol{R}+\boldsymbol{R}, s)$ and $(\boldsymbol{C}+\boldsymbol{C}, s)$. As for Lie groups, our methods classify isomorphism classes of symmetric spaces almost as easily as types. The duality for $GB(K, J)$ induces the classical duality for symmetric spaces.

In the case of the fields $(\boldsymbol{R}, 1)$ and $(\boldsymbol{C}, c)$ we can modify the definitions of B and GB by requiring J to be a positive involution—i.e. for all $x \in A$, trace $(J(x)\cdot x) \geqq 0$. This is equivalent (except in a handful of low dimensional cases) to requiring G to be compact. The condition is preserved by tensor products, and we obtain new Brauer groups $B^c(\boldsymbol{C}, c)$, $GB^c(\boldsymbol{C}, c)$, $B^c(\boldsymbol{R}, 1)$ and $GB^c(\boldsymbol{R}, 1)$. The composite maps

$$B^c(\boldsymbol{R}, 1) \to B(\boldsymbol{C}, 1), \quad B^c(\boldsymbol{R}, 1) \to B(\boldsymbol{R}+\boldsymbol{R}, s), \quad B^c(\boldsymbol{C}, c) \to (\boldsymbol{C}+\boldsymbol{C}, s),$$

$$GB^c(\boldsymbol{R}, 1) \to GB(\boldsymbol{C}, 1)\ GB^c(\boldsymbol{R}, 1) \to GB(\boldsymbol{R}+\boldsymbol{R}, s)\ GB^c(\boldsymbol{C}, c) \to GB(\boldsymbol{C}+\boldsymbol{C}, s),$$

are all isomorphisms, and even induce bijections of the sets of equivalence classes of the corresponding algebras. Thus, the bijection $B^c(\boldsymbol{R}, 1) \to B(\boldsymbol{C}, 1)$ corresponds to the standard bijection between compact and complex Lie groups. Also, dualising and then forgetting the grading gives rise to further bijections

$$GB^c(\boldsymbol{R}, 1) \to B(\boldsymbol{R}, 1), \quad GB^c(\boldsymbol{C}, c) \to B(\boldsymbol{C}, c),$$

also valid at the level of isomorphism classes. We thus obtain a long list of natural bijective correspondences: between graded central simple algebras, the same with positive anti-involution, compact symmetric spaces, complex symmetric spaces, symmetric spaces over double fields, contractible symmetric spaces, real Lie groups, and anti-involuted simple algebras. The correspondences of symmetric spaces preserve, inter alia, the property of being Kählerian. Most of these are, of course, well known (see e.g. [**1**], [**4**]).

A final remark about the starting point of the whole investigation. If V is an orthogonal space over a field k, we have a Clifford algebra $C(V)$. Using universal properties, one obtains a unique involution I of $C(V)$ inducing -1 on V, and a unique anti-involution J inducing $+1$ on V. Thus $C(V)$ determines an element of $GB(k, 1)$; moreover, since $C(V \oplus W) = C(V) \oplus C(W)$, one can easily calculate the element by decomposing V into orthogonal 1-dimensional subspaces. Taking $k = \boldsymbol{R}$, we can write V as a sum of a positive definite space of dimension p, say, and a negative definite space of dimension q. We saw in [**3**] that the class of $C(V)$ in $GB(\boldsymbol{R})$ (or $GB(\boldsymbol{R}+\boldsymbol{R}, s)$) was determined by $(p-q)$ (mod 8). Further, its class in $GB(\boldsymbol{C}, 1)$ is determined by $(p+q)$ (mod 8): to find class in $GB(\boldsymbol{R}, 1)$ we need both. The class in $GB(\boldsymbol{C}, c)$ is determined by p(mod 2) and q(mod 2). Moreover, J is positive if and only if $q=0$ (i.e. V is positive definite); J' only when $p=0$, and V is negative definite. In all other cases, the signatures (where defined) of J and J' vanish.

It is a pleasure to record thanks to the Centro de Investigación y de Estudios Avanzados del IPN, México, for hospitality during the period when the above research was carried out.

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