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REPRESENTATION OF FRACTIONAL POWERS OF INFINITESIMAL GENERATORS OF SEMIGROUPS

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1. Introduction. Let X be a real or complex Banach space and $\mathfrak{E}(X)$ the Banach algebra of endomorphisms of X. Let $\{T(u); u \geqq 0\}$ be an equibounded semigroup of operators of class $(\mathfrak{C}_0)$ in $\mathfrak{E}(X)$ with infinitesimal generator A. A is a closed linear operator with domain $D(A)$ dense in X. For these concepts see e.g. E. Hille-R. S. Phillips [5, Chapters X–XII].

One purpose of this note is to give a representation for the fractional power $(-A)^\gamma$, $\gamma > 0$, of the operator $(-A)$. The result obtained will be a generalization of one due to J. L. Lions-J. Peetre [7, Chapter VII, §2]: *Let γ be a positive integer. An element $f \in X$ belongs to $D((-A)^\gamma)$ if and only if the integral*

$$\frac{1}{C_{\gamma,r}} \int_\epsilon^\infty \frac{[I - T(u)]^r f}{u^{1+\gamma}}\, du \qquad (\epsilon > 0;\ r \text{ any integer} > \gamma). \tag{1}$$

converges in norm as $\epsilon \to 0+$, the constant $C_{\gamma,r}$ being given by

$$C_{\gamma,r} = \int_0^\infty \frac{(1 - e^{-u})^r}{u^{1+\gamma}}\, du. \tag{2}$$

The limit is then equal to $(-A)^\gamma f$.

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We show that this result holds for arbitrary $\gamma>0$. The proof indicated below has its origin in a paper of U. Westphal [8] on fractional derivatives of functions in $L^p(0,\infty)$, $1\leqq p<\infty$, in connection with the semigroup of right translations; the author uses Laplace transform methods (see also H. Berens-U. Westphal [3]).

Of the various methods in defining the fractional power $(-A)^\gamma$, $\gamma>0$, of a closed linear operator A (cf. A. V. Balakrishnan [1], [2], H. Komatsu [6], K. Yosida [9, Chapter IX, §11]) we follow the one by Balakrishnan [2] but restrict ourselves ab initio to the case that A is the generator of the semigroup $\{T(u); u\geqq 0\}$ under consideration. Under these assumptions the resolvent $R(\lambda; A)$ of A exists for each $\lambda>0$ and $\|R(\lambda; A)f\|\leqq M\|f\|$ for all $f\in X$ uniformly with respect to $\lambda>0$, M being some positive constant. For $\gamma>0$ with $k-1<\gamma<k$, $k=1, 2, \cdots$, we define the linear operator J^γ on $D(A^k)$ by

$$J^\gamma f = -\frac{\sin\gamma\pi}{\pi}\int_0^\infty \lambda^{\gamma-k}R(\lambda; A)A^k f d\lambda.$$

J^γ is preclosed and has the usual properties of fractional powers. The operator $(-A)^\gamma$ is then defined to be the smallest closed extension of J^γ. It is easily verified that for each $f\in D(A^k)$ and $k-1\leqq\gamma<k$ ($\gamma=0$ excluded) $(-A)^\gamma f$ has the representation

$$(-A)^\gamma f = \frac{1}{C_{\gamma,k}}\int_0^\infty \frac{[I-T(u)]^k f}{u^{1+\gamma}}du, \tag{3}$$

where $C_{\gamma,k}$ is given by (2).

We only present here our main results. The full details will be published elsewhere.

2. **Fundamental lemma.** The main new tool is

LEMMA. *Let $0<\gamma<r$, $r=1, 2, \cdots$. For all $f\in X$ the integral*

$$\int_0^\infty T(u)f\, q_{\gamma,r}(u)du \tag{4}$$

belongs to $D((-A)^\gamma)$ and

$$(-A)^\gamma\left[\int_0^\infty T(u)f\, q_{\gamma,r}\left(\frac{u}{\epsilon}\right)\frac{du}{\epsilon}\right] = \int_\epsilon^\infty \frac{[I-T(u)]^r f}{u^{1+\gamma}}du \qquad (\epsilon>0), \tag{5}$$

where $q_{\gamma,r}(u)$ is defined by its Laplace transform

$$\int_0^\infty e^{-su}q_{\gamma,r}(u)du = \frac{1}{s^\gamma}\int_1^\infty \frac{(1-e^{-su})^r}{u^{1+\gamma}}du \qquad (\operatorname{Re} s>0).$$

SKETCH OF PROOF. Introducing the auxiliary functions

$$a(u) = \frac{u^{\gamma-1}}{\Gamma(\gamma)} \quad (u > 0), \qquad b_\epsilon(u) = \begin{cases} 0, & 0 < u \leqq \epsilon, \\ u^{-1-\gamma}, & u > \epsilon, \end{cases} \quad (\epsilon > 0),$$

the function $q_{\gamma,r}(u)$ has the representation

$$q_{\gamma,r}(u) = \sum_{i=1}^{r} (-1)^i \binom{r}{i} \left\{ i^\gamma [a * b_i](u) - \frac{1}{\gamma} a(u) \right\}, \tag{6}$$

where $[a * b_i](u)$ denotes the Laplace convolution of $a(u)$ and $b_i(u)$. Note that $q_{\gamma,r}(u)$ belongs to $L^1(0, \infty)$ with $\int_0^\infty q_{\gamma,r}(u)du = C_{\gamma,r}$. Thus the integral (4) exists in the sense of Bochner and

$$\operatorname*{s-lim}_{\epsilon\to 0+} \int_0^\infty T(u)f\, q_{\gamma,r}\left(\frac{u}{\epsilon}\right)\frac{du}{\epsilon} = C_{\gamma,r}f \qquad (f \in X). \tag{7}$$

For any fixed real $\gamma > 0$, choose the integer k such that $k-1 \leqq \gamma < k$ and let the integer $r \geqq k$. Then for each $f \in X$ and $\epsilon, \eta > 0$ the following *fundamental identity* holds:

$$\begin{aligned} &\int_\eta^\infty \frac{[I - T(v)]^k}{v^{1+\gamma}} dv \int_0^\infty T(u)f\, q_{\gamma,r}\left(\frac{u}{\epsilon}\right)\frac{du}{\epsilon} \\ &\qquad = \int_\epsilon^\infty \frac{[I - T(v)]^r}{v^{1+\gamma}} dv \int_0^\infty T(u)f\, q_{\gamma,k}\left(\frac{u}{\eta}\right)\frac{du}{\eta}. \end{aligned} \tag{8}$$

By (7) the right-hand side of (8) is strongly convergent as $\eta \to 0_+$ and

$$\begin{aligned} \operatorname*{s-lim}_{\eta\to 0+} \frac{1}{C_{\gamma,k}} &\int_\eta^\infty \frac{[I - T(v)]^k}{v^{1+\gamma}} dv \int_0^\infty T(u)f\, q_{\gamma,r}\left(\frac{u}{\epsilon}\right)\frac{du}{\epsilon} \\ &= \int_\epsilon^\infty \frac{[I - T(v)]^r f}{v^{1+\gamma}} dv. \end{aligned}$$

Then in case $f \in D(A^k)$, relation (3) gives the result (5) for all $f \in D(A^k)$. Since $D(A^k)$ is dense in X and $(-A)^\gamma$ is a closed operator, (5) finally holds for all $f \in X$.

It remains to prove the identity (8). Using the representation (6) of $q_{\gamma,k}(u)$, the right-hand side of (8) can be rewritten in the form

$$\begin{aligned} \sum_{j=1}^{r} \sum_{i=1}^{k} (-1)^{j+i} \binom{r}{j}\binom{k}{i} j^\gamma &\int_0^\infty [T(v) - I] b_{j\epsilon}(v) dv \\ &\cdot \int_0^\infty T(u)f \left\{ i^\gamma [a * b_{i\eta}](u) - \frac{1}{\gamma\eta^\gamma} a(u) \right\} du. \end{aligned}$$

Setting

$$I_1 \equiv j^\gamma \int_0^\infty T(v) b_{j\epsilon}(v) dv \int_0^\infty T(u) f \left\{ i^\gamma [a * b_{i\eta}](u) - \frac{1}{\gamma \eta^\gamma} a(u) \right\} du$$

and interchanging the order of integration, then

$$I_1 = j^\gamma \int_0^\infty T(u) f du \int_0^u b_{j\epsilon}(v) \left\{ i^\gamma [a * b_{i\eta}](u - v) - \frac{1}{\gamma \eta^\gamma} a(u - v) \right\} dv$$

$$= \int_0^\infty T(u) f \left\{ (ij)^\gamma [a * b_{i\eta} * b_{j\epsilon}](u) - \frac{j^\gamma}{\gamma \eta^\gamma} [a * b_{j\epsilon}](u) \right\} du.$$

Furthermore,

$$I_2 \equiv j^\gamma \int_0^\infty b_{j\epsilon}(v) dv \int_0^\infty T(u) f \left\{ i^\gamma [a * b_{i\eta}](u) - \frac{1}{\gamma \eta^\gamma} a(u) \right\} du$$

$$= \int_0^\infty T(u) f \left\{ \frac{i^\gamma}{\gamma \epsilon^\gamma} [a * b_{i\eta}](u) - \frac{1}{\gamma^2 \epsilon^\gamma \eta^\gamma} a(u) \right\} du.$$

Since the difference $I_1 - I_2$ is symmetric in the pairs (ϵ, j) and (η, i), this proves that the right-hand side of (8) is equal to the left-hand one.

REMARK. The relation (5) and the identity (8) are counterparts of the relation

$$A \left[\frac{1}{u} \int_0^u T(v) f \, dv \right] = \frac{T(u) - I}{u} f, \qquad (u > 0; f \in X),$$

and the identity

$$\frac{T(t) - I}{t} \left[\frac{1}{u} \int_0^u T(v) f \, dv \right] = \frac{1}{t} \int_0^t T(v) \frac{[T(u) - I] f}{u} dv \qquad (t, u > 0; f \in X)$$

for the operator $(-A)^\gamma$. These two simple equations are cornerstones of semigroup theory.

3. Representation theorems.

Let us now formulate some of the many possible applications of the lemma.

THEOREM 1. *Let $0 < \gamma < r$, $r = 1, 2, \cdots$. An element $f \in X$ belongs to the domain of $(-A)^\gamma$ if and only if the integral* (1) *converges strongly as $\epsilon \to 0^+$. In this case we have*

$$(-A)^\gamma f = \operatorname*{s-lim}_{\epsilon \to 0^+} \frac{1}{C_{\gamma,r}} \int_\epsilon^\infty \frac{[I - T(u)]^r f}{u^{1+\gamma}} du.$$

This theorem is the generalization of the result of Lions-Peetre [7] to arbitrary $\gamma>0$. We note that the proof given by the latter authors in their particular instance is entirely different from ours and uses distribution theory.

COROLLARY. *If X is reflexive, then the condition that the integral* (1) *converges strongly for $\epsilon\to 0^+$ may be replaced by*

$$\left\| \int_\epsilon^\infty \frac{[I - T(u)]^r f}{u^{1+\gamma}}\, du \right\| = O(1) \qquad (\epsilon \to 0_+).$$

The proof follows by a result of P. L. Butzer (see [4, §2.1]).

THEOREM 2. *An $f\in X$ belongs to $D((-A)^\gamma)$, $k-1<\gamma<k$, $k=1, 2, \cdots$, if and only if*

$$\operatorname*{s-lim}_{\epsilon\to 0^+} \frac{1}{\Gamma(-\gamma)} \int_\epsilon^\infty \left\{ T(u)f - \sum_{i=0}^{k-1} \frac{u^i}{i!} A^i f \right\} \frac{du}{u^{1+\gamma}}$$

exists, the limit then being equal to $(-A)^\gamma f$.

This is a generalization of a result in [1].

Let X^* be the dual space of X and $\{T^*(u); u\geqq 0\}$ the dual semi-group in $\mathfrak{E}(X^*)$. Set

$$X_0^* = \{ f^* \in X^*; \lim_{u\to 0^+} \|T^*(u)f^* - f^*\| = 0\}.$$

Concerning fractional powers of the dual operator $(-A^*)$ of $(-A)$ we have

THEOREM 3. *Let $0<\gamma<r$, $r=1, 2, \cdots$. For an $f^*\in X^*$ the following assertions are equivalent:*

(i) *$f^*\in D((-A^*)^\gamma)$.*

(ii) *$f^*\in X_0^*$ and the weak* limit of*

$$\frac{1}{C_{\gamma,r}} \int_\epsilon^\infty \frac{[I^* - T^*(u)]^r f^*}{u^{1+\gamma}}\, du$$

exists as $\epsilon\to 0_+$, the limit being equal to $(-A^*)^\gamma f^*$.

(iii) *$f^*\in X_0^*$ and*

$$\left\| \frac{1}{C_{\gamma,r}} \int_\epsilon^\infty \frac{[I^* - T^*(u)]^r f^*}{u^{1+\gamma}}\, du \right\| = O(1) \qquad (\epsilon \to 0_+).$$

The equivalence (iii) depends upon a result due to K. de Leeuw (see [4, §2.1]).

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