THE BOUNDED LINEAR OPERATORS THAT COMMUTE WITH THE BERNSTEIN OPERATORS

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Let $C[0, 1]$ denote the linear space of real-valued continuous functions on $[0, 1]$ normed by

$$\|f\| = \max_{0 \leq x \leq 1} |f(x)|$$

and P_n the subspace of $C[0, 1]$ consisting of the polynomials of degree $\leq n$. For each n ($n=0, 1, 2, \cdots$) we denote by B_n the operator

$$B_n: C[0, 1] \rightarrow P_n$$

defined by

$$(B_nf)(x) = \sum_{k=0}^{n} f(k/n) \binom{n}{k} x^k(1-x)^{n-k} \qquad (n \geq 1),$$

$$(B_0f)(x) = f(0).$$

We call B_n *the Bernstein operator of order n* and B_nf *the nth Bernstein polynomial of f*. The purpose of this note is to characterize those bounded linear operators T

$$T: C[0, 1] \rightarrow C[0, 1]$$

which satisfy

$$(1) \qquad TB_n = B_nT \qquad (n = 0, 1, 2, \cdots)$$

on $C[0, 1]$. We observe that it is sufficient to require (1) to hold on $P = \bigcup_n P_n$ since P is dense in $C[0, 1]$.

LEMMA 1. (a) $B_nf=0$ *if and only if*

$$f \in N_n = \{g \in C[0, 1]: g(k/n) = 0, \quad 0 \leq k \leq n\}, \quad (n \geq 1),$$
$$(N_0 = \{g \in C[0, 1]: g(0)=0\}).$$

(b) B_n *is onto* P_n.

LEMMA 2. *If* T *satisfies* (1) *on* P *then*
(a) $T: P_n \rightarrow P_n$;
(b) $f \in N_n$ *implies* $Tf \in N_n$.

PROOF. $B_nTf \in P_n$ and hence $TB_nf \in P_n$. Since B_n is onto P_n, (a) is established. Next, suppose that $f \in N_n$. Then $TB_nf=0$ and hence $B_nTf=0$, which implies that $Tf \in N_n$.

DEFINITION. Let $\{p_k: k=0, 1, 2, \cdots\}$ be the family of polynomials

$$p_k(x) = 1 \quad \text{if } k = 0,$$
$$= x \quad \text{if } k = 1,$$
$$= \prod_{i=0}^{k-1}\left(x - \frac{i}{k-1}\right) \quad \text{if } 2 \leqq k < \infty.$$

LEMMA 3. (a) *The* $\{p_k: k=0, 1, 2, \cdots\}$ *are linearly independent and each* $p \in P_n$ *admits a unique expansion* $p=a_0p_0+a_1p_1+\cdots+a_np_n$.

(b) *If* $B_np_k=\sum_{s=0}^{n} C_s(n, k)p_s$ *then*

(i) $C_s(n, k)=0$ if $s \not\equiv k \pmod 2$,

(ii) $C_0(n, k)=C_1(n, k)=0$ *if* $2 \leqq k < \infty$,

(iii) $C_s(n, k)=0$ *if* $s > k$.

PROOF. (a) is obvious. To prove (b, i) we observe that for $k \neq 1$

$$(-1)^k(B_np_k)(x) = (B_np_k)(1 - x) = \sum_{s=0}^{n} C_s(n, k)(-1)^s p_s(x) + C_1(n, k).$$

For $2 \leqq k < \infty$ we have $(B_np_k)(0)=p_k(0)=0$ so that $C_0(n, k)=0$. In addition $(B_np_k)(1)=p_k(1)=0$ $(2 \leqq k < \infty)$ from which we may conclude that $C_1(n, k)=0$ $(2 \leqq k < \infty)$. Finally, the image of P_k under B_n is just P_k for $k \leqq n$, and this gives (b, iii).

DEFINITION. Define the operators U_0, U_1, U and $\tilde{U}$ by

$$(U_0f)(x) = f(0), \qquad (U_1f)(x) = (f(1) - f(0))x,$$
$$(Uf)(x) = \tfrac{1}{2}(f(x) + f(1 - x)), \qquad \tilde{U} = I - U.$$

LEMMA 4. *For each* n, $n=0, 1, 2, \cdots$,

$$U_0B_n = B_nU_0, \qquad U_1B_n = B_nU_1,$$
$$UB_n = B_nU, \qquad \tilde{U}B_n = B_n\tilde{U}.$$

PROOF. $(U_0B_nf)(x)=(B_nf)(0)=f(0)=(B_nU_0f)(x)$. $(U_1B_nf)(x) =((B_nf)(1)-(B_nf)(0))x=(f(1)-f(0))x=(B_nU_1f)(x)$. Finally

$$(B_nUf)(x) = \sum_{k=0}^{n} \frac{f(k/n) + f(1 - k/n)}{2}\binom{n}{k}x^k(1 - x)^{n-k}$$
$$= \sum_{k=0}^{n} f(k/n)\binom{n}{k}\frac{x^k(1 - x)^{n-k} + (1 - x)^k x^{n-k}}{2}$$
$$= (UB_nf)(x).$$

The result for $\tilde{U}$ now follows, since $\tilde{U}=I-U$.

THEOREM. *A necessary and sufficient condition that a bounded linear operator* T: $C[0, 1]\rightarrow C[0, 1]$ *satisfy* (1) *is that*

(2) $T = a_0U_0 + a_1U_1 + aU(I - U_0 - U_1) + \tilde{a}\tilde{U}(I - U_0 - U_1)$.

PROOF. (i) By Lemma 4 it follows that any operator of the form (2) satisfies (1).

(ii) By Lemma 2(a) we have $Tp_0=\sigma_0p_0$ and by Lemma 2(a, b) $Tp_k=\sigma_kp_k$ for $1\leqq k<\infty$. By Lemma 3(a) this determines T on P. It suffices to determine the $\{\sigma_k\}$ such that $TB_np_k=B_nTp_k$ for all k and n. Now

$$TB_np_0 = Tp_0 = \sigma_0p_0 = B_nTp_0 \qquad (n = 0, 1, 2, \cdots),$$
$$TB_np_1 = Tp_1 = \sigma_1p_1 = B_nTp_1 \qquad (n = 1, 2, \cdots),$$

while $TB_0p_1=0=B_0Tp_1$. Thus σ_0 and σ_1 may be chosen arbitrarily. Henceforth assume that $2\leqq k<\infty$. Then

$$TB_np_k = \sum_{s=2}^{n} C_s(n, k)\sigma_sp_s$$

while

$$B_nTp_k = \sum_{s=2}^{n} C_s(n, k)\sigma_kp_s$$

so that $C_s(n, k)(\sigma_s-\sigma_k)=0$ $(2\leqq s\leqq n,\ 2\leqq k<\infty,\ n=0,\ 1,\ 2, \cdots)$. If we take $n=2$, then $B_np_k\neq 0$ for $k\equiv 0 \pmod 2$ and hence $C_2(2, k)\neq 0$ for $k\equiv 0 \pmod 2$. Thus $\sigma_k=\sigma_2$ for $k\equiv 0 \pmod 2$. Assume next that $\sigma_3=\sigma_5= \cdots =\sigma_{2j+1}$. Then by Lemma 3(b, i)

$$B_{2j+1}p_{2j+3} = \sum_{\substack{0\leq s\leq 2j+1;\\ s\equiv 1 \pmod 2}} C_s(2j + 1, 2j + 3)p_s.$$

Since $B_{2j+1}p_{2j+3}\neq 0$ there exists a j_0, $0<j_0\leqq j$ such that $C_{2j_0+1}(2j+1, 2j+3)\neq 0$ and this implies that $\sigma_{2j+3}=\sigma_{2j+1}$. Hence

$$\sigma_k = \sigma_2 \quad \text{if } k \equiv 0 \pmod 2\ 2 \leqq k < \infty,$$
$$\sigma_k = \sigma_3 \quad \text{if } k \equiv 1 \pmod 2\ 2 \leqq k < \infty,$$

so that if $p=\sum_{s=0}^{n} b_sp_s$ then

$$Tp = \sigma_0b_0p_0 + \sigma_1b_1p_1 + \sigma_2 \sum_{2\leq s\leq n;\, s\equiv 0 \pmod 2} b_sp_s + \sigma_3 \sum_{2\leq s\leq n;\, s\equiv 1 \pmod 2} b_sp_s.$$

Finally $b_0=p(0)$ and $b_0+b_1=p(1)$ and thus

$$Tp = \sigma_0U_0p + \sigma_1U_1p + \sigma_2U(I - U_0 - U_1)p + \sigma_3\tilde{U}(I - U_0 - U_1)p$$

satisfies (2) on P. But P is dense in $C[0, 1]$ and hence T satisfies (2) on $C=[0, 1]$.

REMARK. The same methods lead to a somewhat more general result. Suppose that for each n, $n=0, 1, 2, \cdots$ X_n denotes the set $\{x_0^{(n)}, x_1^{(n)}, \cdots, x_n^{(n)}\}$ of distinct points on $[0, 1]$, with the properties that for $n \geqq 3$ there exists some i such that $X_i \not\subseteq X_n$ and that $x_0^{(n)}=0$ $(n=0, 1, 2, \cdots)$. Let B_k $(k=0, 1, 2, \cdots)$ be a bounded linear operator B_k: $C[0, 1] \to P_k$ $(k=0, 1, 2, \cdots)$ satisfying:

(1) $B_k 1 = 1 (k=0, 1, 2, \cdots)$, $B_k x = x (k=1, 2, \cdots)$, $B_0 f = f(0)$.

(2) $B_k f = 0$ if and only if

$$f \in N_k = \{g \in C[0, 1]: g(x_k^{(j)}) = 0, \quad 0 \leqq j \leqq k\}.$$

(3) $B_k S = S B_k$ $(k=0, 1, 2, \cdots)$ where $(Sf)(x) = f(1-x)$.

Then the Theorem holds with an identical proof which we omit. In addition to the Bernstein operators, the (Lagrange) interpolation operators, L_k, defined by

$$(L_k f)(x) = \sum_{j=0}^{k} f(x_j^{(k)}) \prod_{\substack{i=0;\\ i\neq j}}^{k} \frac{x - x_i^{(k)}}{x_j^{(k)} - x_i^{(k)}}$$

are included in the more general result, provided that the interpolating sets X_k are invariant under the transformation $x \to 1-x$.

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