RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited. Manuscripts more than eight typewritten double spaced pages long will not be considered as acceptable.

THE INVERSION ENUMERATOR FOR LABELED TREES

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1. One of us (C.L.M.), examining the cumulants of the lognormal probability distribution, noticed that they involve certain polynomials $J_n(x)$ of degree $\frac{1}{2}n(n-1)$, which suggests inversions (the number of inversions of a permutation is the number of transpositions needed to restore the standard order), and with $J_n(1)=n^{n-2}$, which suggests labeled trees. And indeed $J_n(x)$ is the enumerator of trees with n labeled points by number of inversions, when inversions are counted in the following way. First, the point labeled 1 is taken as a root. Then inversions are counted on each branch, ordered away from the root; the number of inversions contributed by a point labeled i on a branch or subbranch is the number of points more remote from the root with labels less than i. It will be shown that

$$J_{n+1}(x) = Y_n(K_1(x), \cdots, K_n(x)) \tag{1}$$

with $K_i(x)=(1+x+\cdots+x^{i-1})J_i(x)$, Y_n the (E.T.) Bell multivariable polynomial, and that

$$\exp \sum_{n=1} \frac{y^n}{n!}(x-1)^{n-1}J_n(x) = \sum_{n=0} \frac{y^n}{n!} x^{C_{n,2}}. \tag{2}$$

To see the connection with the lognormal distribution, suppose ξ is a normal random variable with mean μ, variance σ^2. Then $\eta=\exp \xi$ is lognormal with $E(\eta^k)=\exp(k\mu+\frac{1}{2}k^2\sigma^2)$, so that the cumulant generating function for η is

$$\log E(\exp t\eta) = \log \sum_{k=0} \frac{t^k}{k!}\exp(k\mu+\tfrac{1}{2}k^2\sigma^2) = \log \sum_{k=0} \frac{y^k}{k!} x^{C_{k,2}}$$

where $y=t\exp(\mu+\frac{1}{2}\sigma^2)$, $x=\exp \sigma^2$. On expanding the right hand side in powers of y, one finds that the coefficient of y^n has a factor $(x-1)^{n-1}$, so attention becomes focused on the polynomials $J_n(x)$ appearing in (2).

2. It is convenient to begin with the remark that the formula for the enumerator $R(y)$ of labeled rooted trees, with

$$R(y) = R_1 y + R_2(y^2/2!) + \cdots + R_n(y^n/n!) + \cdots$$

and R_n the number of rooted trees with n labeled points, given by George Pólya in his famous paper [1] on trees, namely

$$R(y) = y \exp R(y)$$

is equivalent to

$$y^{-1}R(y) = \sum_{n=0} \frac{y^n}{n!} Y_n(R_1, \cdots, R_n)$$

or

(3) $$T_{n+1} = (n+1)^{-1}R_{n+1} = Y_n(R_1, \cdots, R_n).$$

If $Y_n = \sum Y_{nk}$, with Y_{nk} the terms of degree k, then Y_{nk} is a representation of trees in which the point 1 is of degree k. In particular, $Y_{n1} = R_n$ represents the planted trees (with $n+1$ points) whose root has the label 1. It follows that if $\bar{J}_n(x)$ is the enumerator of such planted trees by number of inversions, then

(4) $$J_{n+1}(x) = Y_n(\bar{J}_1(x), \cdots, \bar{J}_n(x)).$$

The planted trees with i the label on the point adjacent to the root $(i = 2(1)(n+1))$ add $i-2$ inversions to the trees with n points; hence

$$\bar{J}_n(x) = (1 + x + \cdots + x^{n-1})J_n(x)$$

and (4) is the same as (1).

It follows at once from (1) and the exponential generating function for Bell polynomials that

(5) $$\sum_{n=0} \frac{y^n}{n!} J_{n+1}(x) = \exp \sum_{n=1} \frac{y^n}{n!} \frac{(x^n - 1)}{x - 1} J_n(x)$$

or, if $j_n(x) = (x-1)^{n-1}J_n(x)$

(5a) $$\sum_{n=0} \frac{y^n}{n!} j_{n+1}(x) = \exp \sum_{n=1} \frac{y^n}{n!} (x^n - 1) j_n(x).$$

Using umbral notation $(a^n(x) = a_n(x),\ a^0(x) = 1)$, write

(6) $$\exp ya(x) = \exp \sum_{n=1} \frac{y^n}{n!} j_n(x).$$

Differentiating with respect to y yields

$$a(x) \exp ya(x) = \exp ya(x) \cdot \sum_{n=0} \frac{y^n}{n!} j_{n+1}(x) \tag{7}$$

or, using (5a),

$$a(x) \exp ya(x) = \exp xya(x)$$

and

$$a_{n+1}(x) = x^n a_n(x) = x^{C_{x+1,2}} \tag{8}$$

since $a_0(x) = 1$.

Note that (6) implies

$$a_n(x) = x^{C_{n,2}} = Y_n(j_1(x), \cdots, j_n(x))$$

whose inverse, by the relation of moments to cumulants, is

$$j_n(x) = Y_n(fa_1(x), \cdots, fa_n(x)), \qquad f^j \equiv f_j = (-1)^{j-1}(j-1)!.$$

It is also worth noting that (1) and the recurrence for Bell polynomials imply

$$J_{n+1}(x) = \sum_{k=0} C_{n-1,k}(1 + x + \cdots + x^k) J_{k+1}(x) J_{n-k}(x).$$

For concreteness, the first few values of $J_n(x)$, omitting functional arguments, are $J_1 = J_2 = 1$, $J_3 = 2 + x$, $J_4 = 6 + 6x + 3x^2 + x^3$, $J_5 = 24 + 36x + 30x^2 + 20x^3 + 4x^4 + x^5$. Note also that $J_n(1) = T_n = n^{n-2}$ is verified by induction since

$$\begin{aligned} J_{n+1}(1) &= Y_n(J_1(1), 2J_2(1), \cdots, nJ_n(1)) \\ &= Y_n(R_1, R_2, \cdots, R_n) = T_{n+1}. \end{aligned}$$

Similarly, $J_n(0) = (n-1)!$ since

$$Y_n(0!, 1!, \cdots, (n-1)!) = C_n(1, 1, \cdots, 1) = n!$$

with $C_n(t_1, \cdots, t_n)$ the cycle indicator of the symmetric group.

Reference

1. G. Pólya, *Kombinatorische Anzahlbestimmungen für Gruppen, Graphen, und chemische Verbindungen*, Acta Math. 68 (1937), 145–253.

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