EXTENSION OF VALUATION THEORY

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By a valuation on a commutative ring R with 1 we mean a pair (v, Γ) where Γ is an ordered (multiplicative) group with zero adjoined and v is a map from R *onto* Γ satisfying

(1) $v(xy) = v(x)v(y)$ for all $x, y \in R$,

(2) $v(x+y) \leqq \max \{v(x), v(y)\}$ for all $x, y \in R$.

This generalizes the field concept; the insistence on "onto" is what allows us to generalize the main field theorems.

PROPOSITION 1. *Let A be a subring of a ring R, P a prime ideal of A. Then the following are equivalent:*

(1) *For each subring B of R and prime ideal Q of B with $A \subset B$, $Q \cap A = P$, one has $A = B$.*

(2) *For $x \in R \setminus A$ there exists a $y \in P$ with $xy \in A \setminus P$.*

(3) *There is a valuation (v, Γ) on R with*

$$A = \{x \in R \mid v(x) \leqq 1\}, \qquad P = \{x \in R \mid v(x) < 1\}.$$

We call pairs (A, P) satisfying the three equivalent conditions *valuation pairs.*

PROPOSITION 2. *The valuations (v, Γ) and (w, Λ) determine the same valuation pair (A, P) if and only if there is an order isomorphism ϕ of Γ onto Λ such that $w = \phi \circ v$.*

Let the valuation (v, Γ) determine the valuation pair (A, P). Then an ideal $\mathfrak{A}$ of A is called *v-closed* if $x \in \mathfrak{A}$, $y \in R$ and $v(y) \leqq v(x)$ implies $y \in \mathfrak{A}$.

PROPOSITION 3. *The v-closed ideals of A are linearly ordered by inclusion. The v-closed prime ideals are in* 1-1 *correspondence with the isolated subgroups of Γ. If $\phi: \Gamma \rightarrow \Gamma/\Sigma$ is the natural map with Σ an isolated subgroup of Γ, then the v-closed prime ideal corresponding to Σ is the ideal of the valuation pair determined by the valuation $(\phi \circ v, \Gamma/\Sigma)$.*

Independence and dominance of valuations are defined as in [5] and the "same" computational lemmas are obtained.

Let R be a ring extension of a ring K, (v_0, Γ_0) a valuation on K. By an extension of (v_0, Γ_0) to R we mean a valuation (v, Γ) on R and an order isomorphism ϕ of Γ_0 into Γ such that $v(x) = \phi \circ v_0(x)$ for all $x \in K$.

PROPOSITION 4. *A valuation* (v_0, Γ_0) *on* K *has extensions to* R *if and only if* $R\mathfrak{A}\cap K\subset\mathfrak{A}$ *where* $\mathfrak{A}=\{x\in K \mid v_0(x)=0\}$.

For the remainder of this announcement we assume that R is an integral extension of K and (v_0, Γ_0) is a valuation on K. If (v, Γ) is an extension of (v_0, Γ_0) we identify and get $\Gamma_0\subset\Gamma$.

PROPOSITION 5. *The following hold*:

(1) (v_0, Γ_0) *has extensions to* R,

(2) Γ/Γ_0 *is torsion for any extension* (v, Γ) *of* (v_0, Γ_0),

(3) *Given* $x\in R$ *there is an* $x'\in R$ *such that* $v(xx')=1$ *for all extensions* (v, Γ) *of* (v_0, Γ_0) *with* $v(x)\neq 0$.

PROPOSITION 6. *Let* (v_i, Γ_i) *be pairwise independent extensions of* (v_0, Γ_0) *and* α_i *nonzero elements of* Γ_i, $i=1, 2, \cdots, n$. *Then there is an* $x\in R$ *such that* $v_i(c)=\alpha_i$ *for each* i.

For (v, Γ) an extension of (v_0, Γ_0), define e_v to be the index of Γ_0 in Γ and f_v be the rank of A/P over A_0/P_0, where (A, P) is the valuation pair determined by (v, Γ) and (A_0, P_0) the valuation pair determined by (v_0, Γ_0). Let n be the rank of $R/R\mathfrak{A}$ over $K/\mathfrak{A}$, where $\mathfrak{A}=\{x\in K \mid v_0(x)=0\}$.

PROPOSITION 7. *Let* (v_i, Γ_i), $i=1, 2, \cdots, r$, *be extensions of* (v_0, Γ_0) *which determine distinct valuation pairs. Then* $\sum_{i=1}^{r} e_{v_i} f_{v_i} \leqq n$.

Results and definitions when R is a Galois extension of K are almost identical to those for fields as in [5], including the classical.

PROPOSITION 8. $efg\pi^d=n$, *where* $e=e_v$, $f=f_v$ *for any extension* (v, Γ) *of* (v_0, Γ_0); g *is the number of extensions of* (v_0, Γ_0); π *is the characteristic of the residue ring* A_0/P_0 *if this is prime*, 1 *otherwise*; d *is a nonnegative integer*; *and* n *is the number of elements in a Galois group for* R *over* K.

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