IS EVERY INTEGRAL NORMAL?

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1. **Introduction.** For the terminology and notation not explained below the reader is referred to [1] and [2].

Let L be a Riesz space. A positive linear functional $0 \leqq \phi \in L^{\sim}$ is called an *integral* whenever for every decreasing sequence $\{u_n\}$ of nonnegative elements of L, $\inf_n u_n = 0$ implies $\inf_n \phi(u_n) = 0$. A positive linear functional $0 \leqq \phi \in L^{\sim}$ is called *normal* whenever $0 \leqq u_\tau \in L$ and $u_\tau \downarrow 0$ (i.e., for every pair u_{τ_1}, u_{τ_2} there is an element u_{τ_3} such that $u_{\tau_3} \leqq \inf(u_{\tau_1}, u_{\tau_2})$ and $\inf u_\tau = 0$) implies $\inf_\tau \phi(u_\tau) = 0$. Finally a positive integral is called *singular* whenever $0 \leqq \chi \leqq \phi$ and χ is a normal positive linear functional implies $\chi = 0$. Every positive linear functional can be written uniquely as the sum of a normal integral and a singular integral.

From the point of view of the theory of positive linear functionals it seems natural to ask the question. Is every integral normal? The following example will show that the answer to this question is in general negative.

Let L be the Riesz space of all real bounded Borel measurable functions on the unit interval $0 \leqq x \leqq 1$ and let $\phi(f) = \int_0^1 f(x)dx$ be the positive linear functional determined by the Lebesgue integral. Then ϕ is an integral in the sense defined above but ϕ is not normal. In fact ϕ is a singular integral. Furthermore, observe that L is Dedekind σ-complete (i.e., every countable subset which is bounded above has a least upper bound).

The answer to this question turns out to be quite different if we restrict the class of Riesz spaces to be considered to a special class of Riesz spaces namely the class of Dedekind complete Riesz spaces. (A Riesz space L is called Dedekind complete whenever every nonempty subset of L which is bounded above has a least upper bound.) For this class of Riesz spaces we will indicate below that the statement "every integral is normal" is logically equivalent to the statement "every cardinal is nonmeasurable." A cardinal α is called measurable whenever there exists a probability measure on the algebra of all subsets of α such that every countable subset has measure

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zero. Let L be a Riesz space. Then by $c(L)$ we denote the smallest cardinal with the property that the cardinal of every maximal disjoint system of L is bounded by $c(L)$. We have shown that if L is Dedekind complete and $c(L)$ is nonmeasurable then every integral on L is normal. In particular, if $c(L)$ is less than the first inaccessible cardinal, then every integral is normal on L. Since recently it was shown by Tarski that even a great many inaccessible cardinals are nonmeasurable and since the problem whether one can consistently assume the existence of measurable cardinals is still open it follows that except possibly for very exceptional Dedekind complete Riesz space, every integral is normal. It is known, however, that it is consistent with the other axioms of set theory that every cardinal is nonmeasurable.

2. **A discussion of the main result.** If there exists a measurable cardinal α, then it is easy to see that this measure induces on the Dedekind complete Riesz space of real bounded functions on α an integral which is not normal. For the converse we need the following two results.

Although not every integral is normal we were able to show (Theorem 49.1 of [1]) that every integral is normal on a large ideal. To be more precise the following result holds.

THEOREM 1. *Let L be a Riesz space and let $0<\phi\in L^{\sim}$ be a positive linear integral on L. Then there exists an ideal $I_\phi\subset L$ (i.e., I_ϕ is a linear subspace of L and $|g|\leqq|f|$, $f\in I_\phi$ implies $g\in I_\phi$) such that I_ϕ is full, i.e., for every $0\neq f\in L$ there is an element $0<u\in I_\phi$ such that $u\leqq|f|$, and the restriction of ϕ to I_ϕ is normal. Equivalently, if $0<\phi\in L^{\sim}$ is a singular integral, then its null ideal $N_\phi=\{f: \phi(|f|)=0\}$ is full in L.*

In addition to Theorem 1 we shall need the following theorem concerning complete Boolean algebras, which is an easy consequence of Theorem 4.1(h) of [4].

THEOREM 2. *Let B be a complete Boolean algebra and let $A\subset B$ be a nonempty subset of B. Then there exists a disjointed subset $\{a_\sigma: \sigma\in\Sigma\}$ (i.e., $\sigma_1\neq\sigma_2$ implies $a_{\sigma_1}\wedge a_{\sigma_2}=0$) such that* $\sup A=\sup(a_\sigma: \sigma\in\Sigma)$ *and for every σ there is an element $a\in A$ such that $a_\sigma\leqq a$.*

We shall now formulate and sketch a proof of the main result.

THEOREM 3. *Assume there are no measurable cardinals and assume that L is a Dedekind complete Riesz space. Then every integral on L is normal.*

The proof of this result is based on Theorem 1. If L has an integral ϕ which is not normal, then according to Theorem 1, there is no loss in generality to assume that its null ideal N_ϕ is order dense in L. Thus for an element $0<u\in L$ with $\phi(u)=1$ there is an upward directed system $\{u_\tau\}$ of positive elements of N_ϕ such that $u=\sup u_\tau$. Consider the projection P_τ on the band generated by u_τ and the projection P on the band generated by u. Then for all τ, $\phi(P_\tau u)=0$, $\phi(Pu)=\phi(u)=1$ and $\sup P_\tau=P$. Thus, by Theorem 2, there exists a system $(Q_\sigma:\sigma\in\Sigma)$ of mutual disjoint projections such that $\sup Q_\sigma=P$ and for every $\sigma\in\Sigma$ there is an element τ such that $Q_\sigma\leqq P_\tau$. Then it can be shown that card (Σ) is measurable. The measure is defined as follows: $\mu(X)=\phi(\sup Q_\sigma(u):\sigma\in X)$ for every nonempty subset $X\subset\Sigma$ and we define $\mu(\phi)=0$.

An examination of the proof shows, however, that also the following slightly more general result holds.

THEOREM 4. *Let L be a Riesz space which satisfies the following two conditions*: (i) *Every band is a projection band and the Boolean algebra of all projections of L is complete.* (ii) *For every $0<u\in L$, $c(I_u)$ is nonmeasurable, where I_u is the ideal generated by u. Then every integral on L is normal.*

Observe that the Riesz space of all real functions on a set whose cardinal is nonmeasurable and which takes on only finitely many different values has the properties (i) and (ii) but is obviously not Dedekind complete.

3. **An application.** Let X be a nonempty set and let R^X be the Riesz space of all real functions on X. Then Theorem 4 enables us to express the nonmeasurability of X in an equivalent form in terms of the positive linear functionals on R^X. In fact, we can prove the following theorem which extends Theorem 3.1 of [**3**].

THEOREM 5. *The nonempty set X has a nonmeasurable cardinal if and only if for every positive linear functional ϕ on R^X there exist elements $x_1,\cdots,x_n\in X$ and positive constants $a_1,\cdots,a_n$ such that $\phi(f)=\sum_{i=1}^n a_i f(x_i)$ for all $f\in R^X$.*

In addition to Theorem 4 the proof makes essential use of the fact that every positive linear functional on any R^X is an integral. (See [**2**, Example 20.8 in Note VI] and for the corresponding result for the Riesz space of all real continuous functions on an arbitrary topological space we refer to [**1**, Example 50.7 in Note XV a].) Indeed, if a positive linear functional on R^X is not of the desired form, then

$\psi(f) = \phi(f) - \Sigma(f(x)\phi(\{x\}): x \in \Phi)$, $f \in F^X$ and $\Phi = \{x: \phi(\{x\}) \neq 0\}$ is a nonzero integral which vanishes on all singletons, and so X is measurable. For the converse we use Theorem 4 in an essential way.

References

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