A THEOREM ON RANK WITH APPLICATIONS TO MAPPINGS ON SYMMETRY CLASSES OF TENSORS

BY MARVIN MARCUS[1]

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1. **Results.** Let R be a field containing a real closed subfield R_0. The main results of this announcement follow.

THEOREM 1. *Let $A_1, A_2, \cdots, A_p$ be $m \times n$ matrices with entries in an infinite subset Ω of R containing the natural numbers in R_0. Let k be a positive integer and assume that the rank of each A_i is at least k. Then there exist nonsingular matrices E and F with entries in Ω such that every set of k rows (columns) of EA_iF is linearly independent, $i=1, \cdots, p$.*

COROLLARY 1. *If the matrices $A_1, \cdots, A_p$ in Theorem 1 each have rank precisely k then every k-square subdeterminant of EA_iF is nonzero, $i=1, \cdots, p$.*

THEOREM 2. *If $A_1, \cdots, A_p$ are n-square complex hermitian matrices all of rank at least k then there exists a nonsingular matrix E such that every set of k rows (columns) of E^*A_iE is linearly independent.*

In 1933, J. Williamson [1] gave necessary and sufficient conditions for the compounds of two matrices to be equal. The nontrivial part of his result states the following: *if A is a complex matrix of rank r and $r > m$ then $C_m(A) = C_m(B)$ if and only if $A = zB$ where $z^m = 1$.* A result closely connected to Theorem 1 and generalizing the Williamson result can be proved. We state our theorem in an invariant setting.

Thus, let V be an n-dimensional space over the complex numbers, let H be a subgroup of the symmetric group S_m, $m \leqq n$, and let χ be a complex valued character of degree 1 on H. A multilinear function $f(v_1, \cdots, v_m)$ is symmetric with respect to H and χ if $f(v_{\sigma(1)}, \cdots, v_{\sigma(m)}) = \chi(\sigma)f(v_1, \cdots, v_m)$ for all $v_1, \cdots, v_m$ in V and all $\sigma \in H$. Let P be a vector space and f a fixed multilinear function symmetric with respect to H and χ, f: $V \times \cdots \times V \to P$, such that for any multilinear function g, g: $V \times \cdots \times V \to U$, also symmetric with respect to H and χ, there exists a linear h: $P \to U$ that makes the following diagram commutative:

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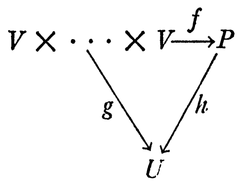
(1) 

Then the pair P, f is called a *symmetry class of tensors* associated with H and χ, e.g., $H=S_m$, $\chi=\text{sgn}$, $P=\bigwedge^m V$, $f(v_1, \cdots, v_m)=v_1\wedge \cdots \wedge v_m$, the usual mth Grassmann product. If T is a linear transformation on V then one defines a linear transformation h via the diagram (1) with $U=P$, $g(v_1, \cdots, v_m)=f(Tv_1, \cdots, Tv_m)$. In this case h is called the transformation *induced* by T and will be denoted here by $K(T)$. If $P=\bigwedge^m V$ then $K(T)$ is the mth compound of T, $C_m(T)$. Another example: if H is the identity group then $P=\bigotimes_{i=1}^m V$, the mth tensor space over V, and $K(T)=\Pi^m(T)$, the mth Kronecker power of T.

We have the following generalization of Williamson's result to an arbitrary symmetry class of tensors as described above. We do not present a proof here but this generalization depends directly on Theorem 1 for the case $p=2$.

THEOREM 3. *If the rank of T is r and $r>m$, then $K(T)=K(S)$ if and only if $T=zS$ where $z^m=1$.*

COROLLARY 2. *If V is a unitary space, the rank of T is r, and $r>m$, then T is normal if and only if $K(T)$ is normal.*

2. **Proof outline.** We say that a set of $m\times n$ matrices $(A_1, \cdots, A_p)$ have property R_k if there exists a nonsingular n-square matrix F such that every set of k columns of A_iF, $i=1, \cdots, p$, is linearly independent: this is abbreviated $(A_1, \cdots, A_p)\in R_k$. It is clear that if we can prove that any set of p matrices all of rank at least k satisfy $(A_1, \cdots, A_p)\in R_k$ then Theorem 1 will follow. Observe that if $S_1, \cdots, S_p$ are nonsingular m-square matrices then

$$(S_1A_1, \cdots, S_pA_p)\in R_k \tag{2}$$

if and only if $(A_1, \cdots, A_p)\in R_k$.

Now let L be the n-square matrix whose (i, j) entry is i^j, $i, j=1, \cdots, n$. It is routine to verify that every subdeterminant of every order of L is nonzero. Next, let $t_1, \cdots, t_n$ be independent indeterminates over R and define an n-square matrix $L(t_1, \cdots, t_n)$ over $R[t_1, \cdots, t_n]$ whose (i, j) entry is $t_i i^j$, $i, j=1, \cdots, n$. It follows that any specialization of $t_1, \cdots, t_n$ to nonzero elements of Ω produces a matrix every one of whose subdeterminants is nonzero. According to (2) we can take $(A_1, \cdots, A_p) = (H_1, \cdots, H_p)$ where H_i is the Hermite normal form of A_i, $i=1, \cdots, p$. Next, consider the matrices $B_i = H_iL(t_1, \cdots, t_n)$ and define the polynomial $p_i(t_1, \cdots, t_n)$ to be the product of all $C_{n,k}$ entries in the first row of the kth compound matrix of B_i, i.e., $C_k(B_i) = C_k(H_i)C_k(L(t_1, \cdots, t_n))$. The fact that A_i and hence H_i has rank at least k implies that there exists a specialization of p_i which is not zero. Hence the polynomial

$$P(t_1, \cdots, t_n) = \prod_{i=1}^{p} p_i(t_1, \cdots, t_n)$$

is not zero. It follows from a standard theorem on polynomials that there exist nonzero elements $\xi_1, \cdots, \xi_n$ in Ω for which $P(\xi_1, \cdots, \xi_n) \neq 0$. In other words, there exist nonzero $\xi_1, \cdots, \xi_n$ in Ω for which every entry in the first row of each of $C_k(H_iL(\xi_1, \cdots, \xi_n))$ is nonzero, $i=1, \cdots, p$. This means that every set of k columns of each of $H_iL(\xi_1, \cdots, \xi_n)$ is linearly independent and proves the result.

The rest of the results announced above follow from Theorem 1.

Reference

1. J. Williamson, *Matrices whose sth compounds are equal*, Bull. Amer. Math. Soc. **39** (1933), 108–111.

University of California, Santa Barbara