

COMPLEX STIEFEL MANIFOLDS, SOME HOMOTOPY GROUPS AND VECTOR FIELDS¹

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Introduction. Throughout this note M will denote a \mathcal{C}^∞ , closed, compact, connected, $2n$ -dimensional, almost-complex manifold. That is, M is an orientable manifold and there is a reduction of the structure group of the tangent bundle of M from $SO(2n)$ to $U(n)$.

Let real (complex) span (M) denote the maximal number of vector fields on M which are linearly independent over the real (complex) numbers. We obtain certain lower bounds on real (complex) span (M).

Our method is that of obstruction theory in the universal example for such a problem, namely a fibration of the form $W_{n+k,k} \rightarrow BU(n) \rightarrow BU(n+k)$. Here $W_{n+k,k}$ is the Stiefel manifold of complex k -frames in complex $(n+k)$ -space, and $BU(l)$ is the classifying space for complex l -plane bundles. We use the results of Mahowald [2] and Thomas [5] in the obstruction-theoretic problems encountered. The work is done in the category of vector bundles over cw -complexes and is then specialized to the case of manifolds.

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Homotopy groups. We give tables of homotopy groups of $W_{n+k,k}$ useful in doing the obstruction theory. More homotopy groups are given than are actually needed.

It is a classically known fact that $W_{n+k,k}$ is $2n$ -connected. Combining this with the fact that, for $p \leq 2(k-1)$ a positive integer, $\pi_{2n+p+1}(W_{n+k,k}) \approx \pi_{2n+p+1}(W_{n+k+l,k+l})$ for $l \geq 1$ an integer, we define $\pi_{2n+p+1}^p(W_{n+k,k})$ to be the p -stem of the complex Stiefel manifolds, for p as above. We obtain essentially the first seven of these stems and some results on the 8 and 9-stems. Some unstable results appear in the tables for the sake of completeness. We remark that $W_{n+1,1} = S^{2n+1}$ and $W_{n,n} = U(n)$ and are not included in the tables, since the homotopy groups in our range of dimensions are well-known for these spaces. The tables list $\pi_{2n+p+1}^p(W_{n+k,k})$:

¹ These results are part of the author's doctoral thesis submitted to the University of California, Berkeley.

$P = 0$	$k \geq 2$	$P = 1$	$k \geq 2$	$P = 2$	$k \geq 2$
$n \geq 1$	Z	n odd	0	n odd	Z
		n even	Z_2	n even	$Z \oplus Z_2$

$P = 3$	$k = 2$	$k \geq 3$
$n = 1$	Z_6	0
2	$Z_{24} \oplus Z_2$	Z_2
n odd ≥ 3	Z_{12}	
n even ≥ 4	$Z_{24} \oplus Z_2$	

and for $k \geq 3, n \geq 3$ we have:

$$\begin{aligned}
 {}^q\pi_{2n+4}(W_{n+k,k}) &= 0 && \text{for any prime } q \geq 5, \\
 {}^3\pi_{2n+4}(W_{n+k,k}) &= 0 && \text{if } n \not\equiv 0 \pmod{3}, \\
 &= Z_3 && \text{if } n \equiv 0 \pmod{3}, \\
 {}^2\pi_{2n+4}(W_{n+k,k}) &= 0 && \text{if } n \equiv 3 \text{ or } 7 \pmod{8} \\
 &= Z_2 && \text{if } n \equiv 1, 2 \text{ or } 6 \pmod{8} \\
 &= Z_4 && \text{if } n \equiv 4 \text{ or } 5 \pmod{8} \\
 &= Z_8 && \text{if } n \equiv 0 \pmod{8}
 \end{aligned}$$

where ${}^q\pi$ is the q -primary part of π .

$P = 4$	$k = 2$	$k \geq 3$
$n = 1$	0	Z
2	$(Z_2)^2$	$Z \oplus Z_2$
n odd ≥ 3	0	Z
n even ≥ 4	Z_2	Z

$P = 5$	$k = 2$	3	$k \geq 4$
$n = 1$	Z_{12}	Z_{24}	0
2	$Z_{24} \oplus Z_2$	Z_{24}	Z_3
$n \text{ odd } \geq 3$	Z_{24}	$Z_{24} \oplus Z_2$	
$n \text{ even } \geq 4$	Z_{24}	Z_{12}	

and, for $n \geq 3, k \geq 4$ we have $\pi_{2n+6}(W_{n+k,k}) \approx \pi_{2n+6}(W_{n+4,3})$ and is given by the table for $P=3$.

$P = 6$	$k = 2$	3	$k \geq 4$
$n = 1$	Z_3	Z_2	Z
2	Z_2	0	Z
3	Z_2	Z_2	
$n \geq 4$	Z_2	$Z_2 \text{ or } 0$	

and, for $n=3, k \geq 4$ the group is Z or $Z \oplus Z_2$. For $n \geq 4, k \geq 4$ we have

$$\begin{aligned} \pi_{2n+7}(W_{n+k,k}) &= Z \quad \text{if } n \neq 4s \\ &= Z \quad \text{or } Z \oplus Z_2 \quad \text{if } n = 4s. \end{aligned}$$

$P = 7$	$k = 2$	$P = 8$	$k = 2$
$n = 1$	Z_{30}	$n = 1$	Z_4
2	Z_{30}	2	$Z_2 \oplus Z_2$
3	Z_{120}	3	$Z_4 \oplus Z_2$
$n \geq 4$	Z_{240}	4	$Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$
		$n \text{ odd } \geq 5$	Z_4
		$n \text{ even } \geq 6$	$Z_2 \oplus Z_2 \oplus Z_2$

For the case $P=0$ the group is in Steenrod [4]. In the case $P=3, k \geq 3, n \geq 3$ the groups follow from the work of Matsunaga [3]. In general, the techniques of Toda, supplemented by information obtained through the use of secondary cohomology operations, give the groups displayed.

Vector fields. Let $C_i(M)$ denote the i th Chern class of the tangent bundle of M , where M is a manifold with the properties described above.

THEOREM 1. *Suppose $H_2(M; Z)$ has no 2-torsion and that n is even, $n \geq 6$, and $\pi_1(M) = 0$. Then, if $C_i(M) = 0$ for $i = n, (n-1)$ and $(n-2)$, real (complex) $\text{span}(M) \geq 6$ (3).*

We remark that this theorem was simultaneously proved by Thomas [6] using different methods. We obtain also

COROLLARY. *Suppose M satisfies the hypotheses of Theorem 1, except that M is not necessarily simply connected. If $n = 12s + 2$ or $12s + 10$ then real (complex) $\text{span}(M) \geq 6$ (3).*

PROOF. The extra obstruction lines in $H^{2n-1}(M; \pi_{2n-2}(W_{n,3}))$. But it is dead there, since $\pi_{2n-2}(W_{n,3}) = 0$, by the above tables in the case $P = 3$.

THEOREM 2. *Suppose M is simply connected, $H_2(M; Z)$ has no torsion, $H_4(M; Z)$ has no 2-torsion, $H_3(M; Z_i) = 0$ for $i = 2$ or 3 and n is odd, $n \geq 9$. Then if $C_i(M) = 0$ $(n-3) \leq i \leq n$, real (complex) $\text{span}(M) \geq 8$ (4).*

THEOREM 3. *Suppose $C_i(M) = 0$ for $(n-4) \leq i \leq n$, n even, $n \geq 8$ and $n \not\equiv 0 \pmod{8}$. Let $\pi_1(M) = 0 = H_3(M; G) = H_5(M; G)$ for $G = Z_2$ or Z_3 . Suppose also that $H_i(M; Z)$ has no torsion for $i = 2, 4$ or 6. Then real (complex) $\text{span}(M) \geq 10$ (5).*

Details of the proofs of these theorems can be found in [1], as well as tables of obstructions. Also larger tables of homotopy groups are given with generators for these groups given in terms of the standard generators for the homotopy groups of spheres.

REFERENCES

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