DUALITY AND ORIENTABILITY IN BORDISM THEORIES

BY DUANE O'NEILL[1]

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1. **Introduction.** A Poincaré duality theorem appears in the literature of bordism theory in several places e.g. [1], [4]. In certain $K(\pi)$-theories, i.e. classical (co)homology theories, the connection between orientability of the tangent bundle of a manifold and this duality is well known [5]. It is interesting to see how this same relationship holds in $\boldsymbol{MG}$-theories and that a simultaneous proof can be given for several different G.

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2. **Notation.** Throughout this note G_n will be one of $O(n)$, $SO(n)$, $U(n)$ or $SU(n)$. We let $\theta=\theta(G_n)$ be the disk bundle associated to the universal G_n-bundle. The Thom space, MG_n, is the total space of θ with the boundary collapsed to a point, the basepoint of MG_n. The Whitney sum of G-disk bundles induces the maps necessary to define the Thom spectrum $\boldsymbol{MG}$ and the maps giving the (co)homology products. We will denote by $(G^*(\ ))G_*(\ )$ the (co)bordism theory associated to $\boldsymbol{MG}$ as in the classical work of G. W. Whitehead [6].

Let dn be the real dimension of the fiber of θ. The inclusion of a fiber into the total space of θ can be thought of as a bundle map covering the inclusion of the basepoint into the classifying space for G_n. There is then the associated map of Thom spaces which we denote by $e_n\colon S^{dn}=D^{dn}/\partial D^{dn}\to MG_n$. If $f\colon S^qX\to MG_n$ is a map, then we denote the associated cohomology class by $(f)\in\tilde{G}^{dn}(S^qX)$. It is easy to prove using the techniques of [6] that (e_n) is the identity element of $\tilde{G}^{dn}(S^{dn})$ and that the identity element

$$e\in\tilde{G}^0(S^0)\xrightarrow{\Sigma^{dn}}(e_n)\in\tilde{G}^{dn}(S^{dn})$$

where Σ^{dn} is the iterated suspension isomorphism.

Now we consider a closed differentiable n-manifold N^n and let $\tau\colon N\to BO(2(n+k))$ be the map classifying the stable unoriented tangent bundle of N. There is the sequence

$$BSU(n+k)\to BU(n+k)\to BSO(2(n+k))\to BO(2(n+k)).$$

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Let τ_G be the lift of τ to the appropriate classifying space, if possible. Let X be a topological space; $f: N \to X$ a map. The set of all triples (N, τ_G, f) forms a monoid under disjoint union. We factor by the relation: $(N, \tau_G, f) = 0$ iff there exists W^{n+1} a differentiable manifold, a τ_G' for W and an $f': W \to X$ such that (i) $\partial W = N$, (ii) $\tau_G' | \partial = \tau_G$, (iii) $f' | \partial = f$. It can be verified that the resulting group is isomorphic to $G_n(X)$ e.g. [1], [2].

3. **Thom classes.** Now we define classes $V_n \in \tilde{G}^{dn}(MG_n)$ by $V_n = (I_n)$ where $I_n: MG_n \to MG_n$ is the identity map and we state the

THEOREM 1. *V_n is an **MG**-orientation class for θ.*

PROOF. We must show that $e_n^*(V_n) \in \tilde{G}^{dn}(S^{dn})$ is the suspension of $e \in \tilde{G}^0(S^0)$. But it is obvious that $e_n^*(V_n) = (e_n \circ I_n) = (e_n)$.

It follows from [3] that the cohomology theory G^* has a Thom isomorphism and a Gysin sequence for G-plane bundles.

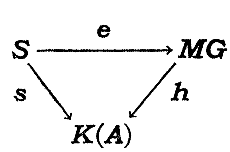
In the remainder of this note, let $A = Z_2$ or Z as appropriate. There exist Thom classes $u_n \in \tilde{H}^{dn}(MG_n, A)$ i.e. θ is $\boldsymbol{K}(\boldsymbol{A})$-orientable. Let $h_n: (MG_n, *) \to (K(A, dn), *)$ be a representative for the homotopy class u_n for every n. We obtain (if $d = 2$, we must extend the collection of maps by some suspensions) a spectral map $\boldsymbol{h}: \boldsymbol{MG} \to \boldsymbol{K}(\boldsymbol{A})$. We notice that the preceding choice of a Thom class in G^*-theory implies that $\boldsymbol{h}$ maps $v_n \to u_n$. Hence, we see the

LEMMA. *$\boldsymbol{h}$ is an isomorphism on the coefficients $\tilde{G}^m(S^m)$.*

REMARK. In [2] Conner and Floyd introduced another spectral map of this type which they call $\beta: U^* \to K_u^*$. Since the definitions of β and $\boldsymbol{h}$ depend on the Thom classes of the range theory, V_n always maps to the nth Thom class. A first Chern class is also defined in [2]. It can be shown that the definition of [2] coincides with the usual definition of first Chern class as $\mathfrak{O}^* \circ j^*(V_1)$, where $\mathfrak{O}$ is the zero section of the bundle and j the map collapsing the total space of the bundle to its Thom space.

4. **Manifolds.** In the remainder of this note, let M be a differentiable manifold of real dimension m, whose stable tangent bundle is a G_l-bundle. We choose representatives $s_n: S^n \to K(A, n)$ for the homotopy class of the generator of $\tilde{H}^n(S^n; A)$ and obtain a spectral map $s: \boldsymbol{S} \to \boldsymbol{K}(\boldsymbol{A})$.

The diagram



commutes and induces the corresponding commutative diagram for the associated cohomology theories. Following [6, p. 271] we write the fundamental cocycle of $H^m(M; A)$ as $\mathbf{s}((j)) = \bar{Z}(M)$ where $j: M \to S^m$ is a map collapsing everything outside a coordinate disk to the basepoint. We call the class $\bar{\sigma}(M) = \mathbf{e}((j)) \in G^m(M)$ the fundamental cocycle as well. Note that $\boldsymbol{h}(\bar{\sigma}(M)) = \bar{Z}(M)$. Recall the definition of [6] that M is $\boldsymbol{MG}$ orientable iff there exists a $t \in G_m(M)$ so that $\langle \bar{\sigma}(M),\ t \rangle = e \in \tilde{G}_0(S^0)$.

THEOREM 2. *M a differentiable manifold with G_l the group of the stable tangent bundle implies that M is $\boldsymbol{MG}$ orientable.*

COROLLARY. *If M as above then M has Poincaré duality with respect to $\boldsymbol{MG}$ (from [6]).*

PROOF OF THEOREM 2. Consider $\sigma(M) = [M, \tau_G, id_M] \in G_m(M)$. One can show that $\boldsymbol{h}(\sigma(M)) = Z(M)$ the fundamental cycle in $H_m(M; A)$ by first showing that $\boldsymbol{h}$ can be identified with an edge homomorphism in a spectral sequence relating G_* and H_* (see [1, p. 17]). Then it is not difficult to calculate the value of $\sigma(M)$.

And finally we have the commutative diagram

$$\begin{array}{ccc} G^m(M) \otimes G_m(M) & \xrightarrow{\langle\ ,\ \rangle} & G_0(pt) \\ \downarrow \boldsymbol{h} \otimes \boldsymbol{h} & & \downarrow \boldsymbol{h} \\ H^m(M; A) \otimes H_m(M; A) & \xrightarrow[\langle\ ,\ \rangle]{} & H_0(pt) \end{array}$$

where the $\boldsymbol{h}$ on the right is an isomorphism.

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UNIVERSITY OF CALIFORNIA, BERKELEY