THE EQUIVALENCE OF SOME GENERAL COMBINATORIAL DECISION PROBLEMS

BY W. E. SINGLETARY[1]

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1. Introduction. A decision problem for a combinatorial system shall denote a pair (Φ, S) where Φ is a specified kind of decision problem (e.g. word problem, halting problem, etc.) and S is a combinatorial system. Likewise, a general combinatorial decision problem, i.e. a decision problem for a class of combinatorial systems, shall denote a pair (Φ, C), where Φ is a specified kind of decision problem and C is a general class of combinatorial systems (e.g. Turing machines, semi-Thue systems, etc.). Clearly, each general combinatorial decision problem $\boldsymbol{P}$ has a class of decision problems for combinatorial systems associated with it. We shall refer to these problems simply as the problems associated with $\boldsymbol{P}$.

There are many papers in the literature which deal with the reduction of one general combinatorial decision problem to another. These papers fall into two general groups. The first group consists of unsolvability proofs such as [1], [8], [10], [11] and [14]. The general format of these proofs is the following: Two general combinatorial decision problems $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are considered, where $\boldsymbol{P}_1$ is known to be unsolvable. Then an effective one-one mapping Ψ of the problems p associated with $\boldsymbol{P}_1$, into the problems associated with $\boldsymbol{P}_2$ and a uniformly effective reduction of p to $\psi(p)$ are given. The second group consists of proofs of the existence of a problem of each r.e. degree of unsolvability associated with some general combinatorial decision problem such as [2], [3], [5], [7], [12] and [13]. The general format of these proofs is the following: Two general combinatorial decision problems $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are considered, where $\boldsymbol{P}_1$ is known to have an associated problem of each r.e. degree of unsolvability. Then an effective one-one mapping ψ of the problems p associated with $\boldsymbol{P}_1$ into the problems associated with $\boldsymbol{P}_2$ and uniformly effective reductions of p to $\psi(p)$ and of $\psi(p)$ to p are given.

Our aim here is to link several of these reductions together in such a way as to provide an effective proof of the equivalence of a number of general combinatorial decision problems. Furthermore, all of our reductions will conform to the second format given above and hence for each pair $\boldsymbol{P}_i$, $\boldsymbol{P}_j$ of general combinatorial decision problems con-

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sidered we shall produce an effective one-one mapping $\psi_{i,j}$ of the problems p associated with P_1 into the problems associated with P_2 such that p is equivalent to $\psi_{i,j}(p)$.

In particular we shall consider general combinatorial decision problems for partial recursive functions, Turing machines, Post normal systems, semi-Thue systems, canonical forms, correspondence classes and propositional calculi.

2. **Preliminary definitions.** If f is a partial recursive function on the nonnegative integers the *definition problem* for f is the problem to determine for an arbitrary nonnegative integer n whether or not $f(n)$ is defined.

If M is a Turing machine the *derivability problem* for M is the problem of determining for arbitrary configurations α and β of M whether or not M started in α will eventually reach β. The *halting problem* for M is the problem of determining of an arbitrary configuration α of M whether or not M started in α eventually halts.

If S is a semi-Thue system, a Post normal system or a canonical form the *word problem* for S is the problem of determining of arbitrary words W_1 and W_2 on the alphabet of S whether or not W_2 is derivable from W_1 in S.

If S_A is a semi-Thue system, Post normal system or canonical form with axiom the decision problem for S_A is the problem of determining of an arbitrary word W on the alphabet of S_A whether or not W is derivable from A in S_A.

A correspondence class C is an effective set of sequences of length n (for some fixed n) of nonempty words over a finite alphabet V. If C is a correspondence class and $\alpha=(\alpha_1, \cdots, \alpha_n)$ and $\beta=(\beta_1, \cdots, \beta_n)$ are sequences of C, then there is a solution for α and β if and only if there is a positive integer l and a finite sequence $i_1, i_2, \cdots, i_l$ of the integers $1, 2, \cdots, n$ such that

$$\alpha_{i_1}\alpha_{i_2} \cdots \alpha_{i_l} = \beta_{i_1}\beta_{i_2} \cdots \beta_{i_l}.$$

The *Post correspondence problem for a correspondence class* C is the problem of determining of arbitrary sequences α and β of C whether nor not they have a solution.

A *correspondence class with axiom* C_α is simply a correspondence class C with a fixed sequence of α of C designated as axiom. The *decision problem for a correspondence class with axiom* C_α is the problem of determining of an arbitrary sequence β of C_α whether or not α and β have a solution.

A *propositional calculus* P is specified by:

(1) *A set S of connectives and a set of propositional variables.* We shall require that S contain at least one binary connective which we shall denote by "$\supset$". The wffs of P are the wffs built up in the usual way from the connectives of S and the propositional variables.

(2) *A set of wffs of P, to be called "axioms."* The theorems of P are those wffs of P which can be derived from the axioms using the two rules of inference:

(i) substitution, and

(ii) $a,\ (a \supset b) \vdash b$.

The *decision problem for a propositional calculus P* is the problem to determine of an arbitrary wff W of P whether or not W is a theorem of P.

3. **The theorem and an outline of the proof.** Let $\boldsymbol{R}$ represent the general definition problem for partial recursive functions, $\boldsymbol{M_D}$ the general derivability problem for Turing machines, $\boldsymbol{M_H}$ the general halting problem for Turing machines, $\boldsymbol{S_W}$ the general word problem for semi-Thue systems, $\boldsymbol{S_D}$ the general decision problem for semi-Thue systems with axiom, $\boldsymbol{N_W}$ the general word problem for Post normal systems, $\boldsymbol{N_D}$ the general decision problem for Post normal systems with axiom, $\boldsymbol{C_W}$ the general Post correspondence problem for correspondence classes, $\boldsymbol{C_D}$ the general decision problem for correspondence classes with axiom, $\boldsymbol{F_W}$ the general word problem for canonical forms, $\boldsymbol{F_D}$ the general decision problem for canonical forms with axiom, and $\boldsymbol{P}$ the general decision problem for propositional calculi.

THEOREM. *The general combinatorial decision problems* $\boldsymbol{R}$, $\boldsymbol{M_D}$, $\boldsymbol{M_H}$, $\boldsymbol{S_W}$, $\boldsymbol{S_D}$, $\boldsymbol{N_W}$, $\boldsymbol{N_D}$, $\boldsymbol{C_W}$, $\boldsymbol{C_D}$, $\boldsymbol{F_W}$, $\boldsymbol{F_D}$ *and* $\boldsymbol{P}$ *are equivalent. Furthermore, for each pair* $\boldsymbol{P_i}$ *and* $\boldsymbol{P_j}$ *of these problems there is an effective mapping* $\psi_{i,j}$ *which when applied to any problem associated with* $\boldsymbol{P_i}$ *will produce an equivalent problem associated with* $\boldsymbol{P_j}$.

COROLLARY. *Every r.e. degree of unsolvability is represented by a problem associated with any of the general combinatorial decision problems of the Theorem.*

We shall indicate how to construct four sequences of reductions which may be linked together to obtain the desired result. These sequences may be represented diagrammatically as follows:

1. $\boldsymbol{R} \xrightarrow{\text{I}} \boldsymbol{M_D} \xrightarrow{\text{II}} \boldsymbol{S_W} \xrightarrow{\text{III}} \boldsymbol{N_W} \xrightarrow{\text{IV}} \boldsymbol{C_W} \xrightarrow{\text{V}} \boldsymbol{R}$

2. $\boldsymbol{R} \xrightarrow{\text{VI}} \boldsymbol{M_H} \xrightarrow{\text{VII}} \boldsymbol{S_D} \xrightarrow{\text{VIII}} \boldsymbol{N_D} \xrightarrow{\text{IX}} \boldsymbol{C_D} \xrightarrow{\text{X}} \boldsymbol{R}$

3. $S_W \xrightarrow{\text{XI}} P \xrightarrow{\text{XII}} F_D \xrightarrow{\text{XIII}} C_D$

4. $N_W \xrightarrow{\text{XIV}} F_W \xrightarrow{\text{XV}} C_W$

where each arrow represents an effective mapping which when applied to any problem associated with the general combinatorial decision problem at the tail of the arrow will produce an equivalent problem associated with the general combinatorial decision problem represented at the head of the arrow. The numbers above the arrows indicate the order in which these reductions will be given.

I. This reduction has been carried out by Shepherdson [**12**]. The idea is to construct a "large scale" machine with derivability problem equivalent to the given definition problem and then perform successive reductions to limited register, single register and finally to Turing machines maintaining the equivalence of the derivability problems at each stage.

II. This reduction is performed in two stages. One first specifies a semi-Thue system from the Turing machine table following Post [**11**]. Left and right symbols are then introduced, as Boone has done in [**2**], and an argument, based on Turing barriers, for the equivalence of the word problem for this system and the derivability problem for the Turing machine can then be made.

III. This reduction has been carried out by Ihrig [**7**]. It is actually a refinement of a reduction given in Davis [**4**].

IV. This reduction is essentially that of Post [**10**]. One can easily verify that such an equivalence reduction is possible by carefully following Post's proof.

V. The details of this reduction have been formally carried out by Cudia and the writer [**3**].

VI. Shepherdson has carried out this reduction in much the same manner as described in I.

VII. The idea here is to first construct a semi-Thue system T_1 whose halting problem is equivalent to the halting problem for the Turing machine. The construction is the same as that described in II. Then, following Davis [**4**], T_1 is altered so as to obtain a second semi-Thue system T_2 having the property that for an arbitrary word, W; T_2 will eventually reach a certain word W_0 if and only if T_1 halts. The antecedent and consequent of each production of T_2 are then interchanged and the word W_0 taken as axiom to obtain the desired system.

VIII. The construction here is essentially the same as that given in III.

IX. The construction here is essentially the same as that given in IV.

X. The reduction given in V can easily be altered to accomplish this.

XI. A proof that this can be done may be found in Gladstone [5], Ihrig [7] or Singletary [13].

XII. That such a reduction as this could be carried out was certainly recognized by Post [8]. The equivalence argument is not difficult.

XIII. The idea here is to first reduce the word problem for the canonical form to that for a Post normal system following Post [8]. Equivalence of the problems may be lost but the first is equivalent to a recursive subset of the second. The word problem for the resulting Post normal system is then reduced to the Post correspondence problem for a correspondence class as in IX. The desired correspondence class is then an effective subset of the resulting one.

XIV. This is trivial since a Post normal system is a canonical form.

XV. The reduction here is similar to that outlined in XIII.

It is perhaps worth noting that each of these sequences ends with a problem for a correspondence class (this is true of 1 and 2 in the diagram since they are circular). The really crucial step in completing these sequences, so far as the writer was concerned, was in noting that although the equivalence of the problems may be lost each time one reduces the problem for a Post normal system to that for a correspondence class, following Post [8], equivalence to the original problem can be maintained by sorting out a recursive subset of the resulting correspondence class.

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THE PENNSYLVANIA STATE UNIVERSITY