CANONICAL FORMS OF CERTAIN VOLTERRA INTEGRAL OPERATORS AND A METHOD OF SOLVING THE COMMUTATOR EQUATIONS WHICH INVOLVE THEM[1]

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The similarity properties of Volterra operators on $L_p[0, 1]$ having reasonably smooth kernels seem to depend entirely on the behavior of the kernel as regards zeros and singularities on the diagonal $x=y$.

If T_G is a Volterra operator on $L_p[0, 1]$, then a study of its similarity properties seems to reduce to the following procedure involving the complex kernel $G(x, y)$.

(1) Classify $G(x, x)$ according to its zeros and singularities on the interval $0 \leqq x \leqq 1$.

(2) Show that T_G is similar to a unique T_p for T_p a canonical kernel of the class of which $G(x, y)$ belongs.

See [1], [2], and [4] for $G(x, y)$ of order $\alpha > 0$ i.e.

$$G(x, y) = (x - y)^{\alpha-1}H(x, y)/\Gamma(\alpha)$$

with $H(x, x) > 0$ and $H(x, y)$ having certain smoothness properties. The canonical form in this case is KJ^α for

$$K = \left[\int_0^1 [H(t, t)]^{1/\alpha} dt \right]^\alpha$$

and

$$J^\alpha f = \int_0^x ((x - y)^{\alpha-1}/\Gamma(\alpha))f(y)dy.$$

See [5] and [6] for $G(x, y)$ of rank 1, i.e.

$$\begin{aligned} G(x, x) &< 0 \qquad \text{if } 0 \leqq x < x_0, \\ G(x, x) &> 0 \qquad \text{if } x_0 < x \leqq 1, \\ G(x_0, x_0) &= 0. \end{aligned}$$

The canonical form in this case is $kQ_{a,\nu}$ for unique real k, a, and ν satisfying $0 \leqq a,\ \nu \leqq 1,\ 0 < k$

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(1) $$\int_0^1 G(t,t)dt = k\int_0^1 (x-a)dx = k/2(1-2a),$$

(2) $$\int_{x_0}^1 G(t,t)dt = k\int_a^1 (x-a)dx = k/2(1-a)^2,$$

(3) $$G_x(x_0, x_0)/(G_x(x_0, x_0) + G_y(x_0, x_0)) = \nu,$$

and

$$Q_{a,\nu}f = \int_0^x (x-a)^\nu (t-a)^{1-\nu} f(t)dt.$$

These canonical forms are unique in a sense made precise in [**4**], [**5**], [**7**] and [**8**].

The important and delicate part of the previous work involved solving an equation of the form

(1) $$Q_{a,\nu}T_{\Gamma(B)} - T_{\Gamma(B)}Q_{a,\nu} = T_B$$

for the kernel $\Gamma(B)$.

This is equivalent to an integral equation

(2) $$\int_y^x [(x-a)^\nu (t-a)^{1-\nu}\Gamma(t,y) - \Gamma(x,t)(t-a)^\nu (y-a)^{1-\nu}]dt = B(x,y).$$

Dupras, in his doctoral thesis [**1**], solved the commutator equation

(3) $$J^\alpha * \Gamma^{(\alpha)}(B) - \Gamma^{(\alpha)}(B) * J^\alpha = B \qquad \text{for all } \alpha > 0$$

using a certain contour integral. He obtained the result

(4) $$\Gamma^{(\alpha)}(B) = \int_0^{x-y} b(\sigma, y)d\sigma - (1/\alpha)\int_0^y b(x-y, t)dt + R_\alpha(x,y)$$

for

(5) $$B(x,y) = \int_0^{x-y} ((x-y-\sigma)^\alpha/\Gamma(\alpha+1))b(\sigma,y)d\sigma,$$

(6) $R_{(\alpha)}(x, y)$ is a certain contour integral depending on α.

Consider the Volterra integral operator

(7) $$Q_{[\alpha,a,\nu,p]} = Q_{[\alpha;\, a_1,a_2,\cdots,a_n;\, \nu_1,\cdots,\nu_n;\, p_1,\cdots,p_n]}$$

which has the kernel $F(x, y)$ such that

(8) $$F(x,y) = ((x-y)^{\alpha-1}/\Gamma(\alpha))G(x,y)$$

with

$$G(x, x) = \prod_{i=1}^{n} (x - a_i)^{p_i}, \qquad 0 \leqq a_1 < a_2 \cdots < a_n \leqq 1, \tag{9}$$

$$G_x(x, x) = \left(\prod_{j=1}^{n} (x - a_j)^{p_j} \right)\left(\sum_{i=1}^{n} \nu_i/(x - a_i) \right). \tag{10}$$

We shall later restrict α, ν_j and p_j in such a way that $Q[\alpha, \mathbf{a}, \boldsymbol{\nu}, \mathbf{p}]$ is always a Volterra integral operator.

Suppose for some complex k, we could get

$$kM_{(1/l)}S_R J^\alpha S_S M_l = Q[\alpha, \mathbf{a}, \boldsymbol{\nu}, \mathbf{p}] \tag{11}$$

for $S_S = f(S(x))$ and $M_l = l(x)f(x)$ both bounded linear invertible transformations on $L_p[0, 1]$ with $S^{-1}(x) = R(x)$ or

$$ku^{-1}J^\alpha u = Q[\alpha, \mathbf{a}, \boldsymbol{\nu}, \mathbf{p}]. \tag{12}$$

It would then follow that the solution to the commutator equation

$$[Q_{[\alpha,\mathbf{a},\nu,\mathbf{p}]}, T_{X^{[\alpha,\mathbf{a},\nu,\mathbf{p}]}}] = T_A$$

is

$$T_{X^{[\alpha,\mathbf{a},\nu,\mathbf{p}]}} = u^{-1}T_{\Gamma}(\alpha)(uT_A u^{-1})u/k$$

or for brevity

$$X^{[\alpha,\mathbf{a},\nu,\mathbf{p}]} = u^{-1}\Gamma^{(\alpha)}(uAu^{-1})u/k. \tag{13}$$

In reality, the transformations u and u^{-1} will, in general, not be bounded, or for that matter well defined. However, we shall use this formalism to obtain candidates for $X^{[\alpha,\mathbf{a},\nu,\mathbf{p}]}$. In addition, the formalism yields a precise definition of $Q_{[\alpha,\mathbf{a},\nu,\mathbf{p}]}$.

The author used this method in [6] and obtained the canonical form for operators of rank one, i.e. the $Q_{a,\nu} = Q_{[1;\, a;\, \nu;\, 1]}$.

The candidate for $X^{[1;\, a;\, \nu;\, 1]}$ was found and was used to obtain the real commutator solution. See [5] and [6] for an exhaustive description of the similarity properties of operators of rank one.

It is known that a Volterra operator T_H with a reasonably smooth kernel $H(x, y)$ commutes with J^α iff $H(x, y) = f(x - y)$.

Thus we would think

$$[T_N, Q_{[\alpha,\mathbf{a},\nu,\mathbf{p}]}] = 0$$

if

$$T_N(x, y) = u^{-1}T_{f(x-y)}u. \tag{14}$$

We may use the operators which commute with $Q_{[\alpha,a,\nu,p]}$ to help make $X^{[\alpha,a,v,p]}$ well defined, i.e.

$$[Q_{[\alpha,a,\nu,p]},\, T_{X[\alpha,a,\nu,p]} + u^{-1}T_{f(x-y)}u] = [Q_{[\alpha,a,\nu,p]},\, T_{X^{[\alpha,a,\nu,p]}}].$$

Now we shall ortain the formal expressions for k, S, l, and $S^{-1}=R$.

$$k(R(x) - R(y))^{\alpha-1}R'(y)(l(y)/l(x)) = (x-y)^{\alpha-1}G(x, y),$$

$$(15)\qquad k[R'(x)]^{\alpha} = G(x, x) = \prod_{i=1}^{n} (x - a_i)^{p_i},$$

$$R(x) = \left(\int_0^x \prod_{i=1}^{n} (t - a_i)^{p_i/\alpha}dt\right)/k^{1/\alpha}$$

with

$$(16)\qquad k = \left[\int_0^1 \prod_{i=1}^{n} (t - a_i)^{p_i/\alpha}dt\right]^{\alpha} \qquad \text{so that } R(1) = 1.$$

(It is possible that $k=0$. We ignore this difficulty and proceed formally.)

If we equate x derivatives at $y=x$, we obtain

$$(17)\qquad l(x) = \prod_{i=1}^{n} (x - a_i)^{(1/2-1/2\alpha)p_i-\nu_i},$$

$$(18)\qquad G(x, y) = \left(\left(\int_y^x \prod_{i=1}^{n} (t - a_i)^{p_i/\alpha}dt\right)/(x-y)\right)^{\alpha-1} \cdot \prod_{i=1}^{n} (x - a_i)^{\nu_i+p_i(1/2\alpha-1/2)}(y - a_i)^{-\nu_i+p_i(1/2\alpha+1/2)}.$$

In order that T_F be a Volterra operator, we require

$$(19)\qquad \alpha \geqq 1, \qquad p_i(1/2 - 1/2\alpha) \leqq \nu_i \leqq p_i(1/2\alpha + 1/2), \qquad 0 < p_i.$$

(This is not the most general case, but is sufficiently general for our purposes here.)

The corresponding commuting operator should be T_N with

$$N(x, y) = f(R(x) - R(y))R'(y)l(y)/l(x),$$

$$(20)\qquad N(x, y) = \prod_{i=1}^{n} (x - a_i)^{\nu_i+p_i(1/2\alpha-1/2)}(y - a_i)^{p_i(1/2\alpha+1/2)-\nu_i} \cdot f\left(\int_y^x \prod_{i=1}^{n} (t - a_i)^{p_i/\alpha}dt\right).$$

It can be shown formally that

$$Q_{[\alpha,a,\nu,p]} * Q_{[\beta,a,\nu,p]} = Q_{[\alpha+\beta,a,\nu,p]}. \tag{21}$$

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