ON THE ERGODIC THEOREM FOR POSITIVE OPERATORS[1]

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Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and let T be a positive linear operator on $L_1(X, \mathcal{A}, \mu)$. The ratio ergodic theorem of Chacon-Ornstein (see [3], [7], [2]) assumes that $|T|_1$, the L_1 norm of T, is less than or equal to one. Here we discuss the behavior of the ratio under the weaker boundedness assumption (b_h). All sets and functions introduced below are assumed measurable. All relations are assumed to hold modulo sets of μ-measure zero. L_1^+ is the class of nonnegative *not identically vanishing* elements of L_1; similar conventions apply to other function spaces. $L_1(A)$ is the class of functions f with supp f (support of f), contained in A and $\int|f| < \infty$. $T^P g$ is the function $g+Tg+T^2g+ \cdots$. The function $f\cdot 1_A$ is sometimes written f_A. A set A is called *closed on* a set B if $f\in L_1^+(A)$ implies $1_B\cdot Tf \in L_1(A)$. A set closed on X is called *closed.*

THEOREM 1. *Let h be a fixed function in L_∞^+ and assume that T satisfies the following condition:*

$$(\mathrm{b_h}) \qquad \sup_n \int T^n f\cdot h < \infty \qquad \textit{for each } f \in L_1^+.$$

Then the space X uniquely decomposes into sets Y^h and Z^h with the following properties. The set Z^h is closed and, if $f\in L_1\ (Z^h)$, then

$$(1) \qquad \gamma^*(f) \overset{\text{def}}{=} \lim_n \left(\sup_j n^{-1} \sum_{i=0}^{n-1} \int (T^{i+j} | f | \cdot h) \right) = 0.$$

If $f\in L_1^+(Y^h)$, then $\gamma^*(f) > 0$.

THEOREM 2. *Assume* $(\mathrm{b_h})$. *The set Y^h decomposes into the conservative part YC^h and the dissipative part YD^h: for each $f\in L_1^+$, $T^P f = 0$ or ∞ on YC^h; $T^P f < \infty$ on YD^h. The subsets of YC^h closed on Y^h form a σ-field, say $\mathcal{C}^h$, and $YC^h \in \mathcal{C}^h$. If $X \neq Z^h$, then the equation*

$$(2) \qquad e \in \overset{+}{L_\infty}, \quad \operatorname{supp} e = Y^h, \quad T^* e = e$$

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admits a solution e which on YC^h *is uniquely determined, modulo multiplication by a* $\mathfrak{C}^h$ *measurable function. If*

$$e\cdot f + f_Z \in \overset{+}{L_1}, \qquad e\cdot g + g_Z \in \overset{+}{L_1},$$

then the ratio

$$D_n(f, g) \overset{\text{def}}{=} \sum_{i=0}^{n-1} T^i f \Big/ \sum_{i=0}^{n-1} T^i g \tag{3}$$

converges to a finite limit on the set $Y \cap \operatorname{supp} T^P g$. *The limit is* $T^P f / T^P g$ *on* $YD^h \cap \operatorname{supp} T^P g$ *and*

$$\frac{E[R(T, YC^h, YD^h) f\cdot e/\mathfrak{C}^h]}{E[R(T, YC^h, YD^h) g\cdot e/\mathfrak{C}^h]} \tag{4}$$

on $YC^h \cap \operatorname{supp} T^P g$, *where*

$$R(T, A, B)f = f_A + (Tf_B)_A + \cdots + (T(T^n f_B)_B)_A + \cdots. \tag{5}$$

(*The conditional expectations in* (4) *are considered as computed with respect to a finite equivalent measure.*)

The standing assumption from now on is (b_1): (b_h) *with* $h=1$, which by the uniform boundedness principle may be stated as:

$$\sup_n |T^n|_1 < \infty. \tag{b$_1$}$$

The superscript $h=1$ *is omitted*: we write Z for Z^1, $\mathfrak{C}$ for $\mathfrak{C}^1$, etc. A stronger statement than Theorem 1 is now true.

THEOREM 3. *If* $f \in L_1(Z)$ *then* $\lim \int T^n f = 0$; *if* $f \in L_1^+(Y)$ *then* $\liminf \int T^n f > 0$.

THEOREM 4. *Assume that* $X = YC$ *and that the σ-field* $\mathfrak{C}$ *is trivial. Then there is a unique, up to multiplicative constants, function e satisfying*:

$$e \in \overset{+}{L_\infty}, \quad \operatorname{supp} e = X, \quad T^* e = e. \tag{6}$$

If $f\cdot e \in L_1^+$, $g\cdot e \in L_1^+$, *then the limit of* $D_n(f, g)$ *is on* X:

$$\frac{\int f\cdot e}{\int g\cdot e} = \lim_n \frac{\sup_j \sum_{i=0}^{n-1} \int T^{i+j} f}{\sup_j \sum_{i=0}^{n-1} \int T^{i+j} g} = \lim_n \frac{\inf_j \sum_{i=0}^{n-1} \int T^{i+j} f}{\inf_j \sum_{i=0}^{n-1} \int T^{i+j} g}. \tag{7}$$

THEOREM 5. *Let $g\in L_1^+(Z)$ and $p\in L_\infty^+(Z)$ be such that*

$$\sum_{i=0}^{\infty}\int T^i g\cdot p = \infty. \tag{8}$$

Then there is a function $f\in L_1^+$ such that

$$\limsup_n \left(\sum_{i=0}^{n-1}\int T^i f\cdot p\right)\Big/\left(\sum_{i=0}^{n-1}\int T^i g\cdot p\right) = \infty. \tag{9}$$

THEOREM 6. *Let $g\in L_1^+(Z)$ be such that $T^P g=\infty$ on Z. Then for each function $p\in L_1^+$ there is a function $f\in L_1^+$ with*

$$\limsup_n \int D_n(f, g)\cdot p = \infty. \tag{10}$$

We now wish to make a statement about the behavior of the ratio at a *point*, and this motivates the following definition. The operator T is called *asymptotically regular* (*regular*) at a point x_0 if for all n sufficiently large (for all positive n), the value of $T^n f$ at x_0 does not depend upon the choice of f in its L_1 equivalence class. In the discrete case T is regular at each point; more generally, operators regular at each point may be defined by *transition measures*. By a *transition measure* we understand a function $T(x, A)$ of two variables which for each fixed $A\in\mathcal{A}$ is a measurable function in $x\in X$; for each fixed $x\in X$, a σ-finite μ-continuous measure in $A\in\mathcal{A}$. A transition measure $T(\ ,\ )$ acts on L_1 by the relation

$$Tf(x) = \int_X T(x, dy)f(y) \qquad f\in L_1. \tag{11}$$

THEOREM 7. *Let T be asymptotically regular at a point $x_0\in Z$ and let $g\in L_1^+(Z)$ be such that $T^P g=\infty$ on Z. Then there is a function $f\in L_1^+$ such that*

$$\limsup_n D_n(f, g)(x_0) = \infty. \tag{12}$$

The following Theorem 8 is concerned with mean convergence to zero. The case $|T|_1=1$ has been independently obtained by Krengel and Neveu (see [**6**]); it is implied by, and implies (cf. [**5**, p. 662]) an L_1 decomposition theorem due to Chacon [**1**]. Mrs. Dowker [**4**] proved Theorem 8 in the case when T is an isometry of L_1 of a probability space, generated by a point transformation τ by the relation $f\circ\tau = T^*f$, $f\in L_\infty$.

THEOREM 8. *Let* $e\in L_\infty^+$ *be such that* supp $e=Y$ *and* $T^*e=e$. *If* $f\in L_1(YC)$ *and* $E(f\cdot e/\mathcal{C})=0$, *then* $n^{-1}(f+Tf+\cdots+T^{n-1}f)$ *converges to zero in* L_1 *mean.*

We now sketch the proof of the main assertions of Theorem 1 and Theorem 2. Let $\{L_\beta, \beta\in B\}$ be the collection of all Banach limits. Assume (b_h). For a fixed $\beta\in B$ we show that there is a T^* invariant function e_β, either null or in L_∞^+, and such that for each $f\in L_1$

$$L_\beta\left(\int T^nf\cdot h\right)=\int f\cdot e_\beta. \tag{13}$$

Define Y^h as the maximal among the sets supp e_β, $\beta\in B$. Theorem 1 now follows because the functional applied in (1) to the sequence $\int T^n|f|\cdot h$ is the maximal value of Banach limits. Among the functions e_β, $\beta\in B$ there is at least one, say e, which is a solution of (2). The operator V defined by

$$Vf=e\cdot T[f/(e+1_{Z_h})]$$

has the L_1 norm less than or equal to one, and the convergence of (3) to a finite limit follows by application to this operator of the Chacon-Ornstein theorem.

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