INNER FUNCTIONS IN POLYDISCS[1]

BY WALTER RUDIN

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For $N=2, 3, 4, \cdots$ the polydisc U^N consists of all $z=(z_1, \cdots, z_N)$ in C^N (the space of N complex variables) such that $|z_j|<1$ for $j=1, \cdots, N$. The class of all bounded holomorphic functions in U^N is denoted by $H^\infty(U^N)$. If $f\in H^\infty(U^N)$ it is well known that the radial limits

$$f^*(z) = \lim_{r\to 1} f(rz) \tag{1}$$

exist for almost all z in the distinguished boundary T^N of U^N. Here $rz=(rz_1, \cdots, rz_N)$.

An *inner function in* U^N is, by definition, a function $g\in H^\infty(U^N)$ such that $|g^*(z)|=1$ for almost all $z\in T^N$.

The present note contains partial answers to questions such as the following: Is every $f\in H^\infty(U^N)$ (other than $f\equiv 0$) a product $f=gh$ where g is inner and both h and $1/h$ are holomorphic in U^N? (In this case we say that f and g *have the same zeros in* U^N.) If not, what are some sufficient conditions on f which guarantee the existence of such a factorization? If f does have the same zeros as some inner function g, does it follow that g can be chosen so that $f/g\in H^\infty(U^N)$?

A special role is played by those inner functions which (for lack of a better name) I propose to call *good*: An inner function g is good if

$$\lim_{r\to 1} \int_{T^N} \log |g(rz)|\, dm(z) = 0. \tag{2}$$

Here dm denotes the Haar measure of T^N.

To see some examples, consider these four classes of inner functions in U^N:

(A) Those which have continuous extensions to the closure of U^N.

(B) Rational inner functions.

(C) Finite or infinite (convergent) products of rational inner functions.

(D) Good inner functions.

In one variable, (A) = (B) and (C) = (D), since the good inner func-

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tions in one variable are precisely the Blaschke products. That (A)$\subset$(B) in the general case is proved in [2]. For $N \geqq 2$, (A)$\neq$(B), as shown by

$$g(z, w) = (4zw - 3z - w)/(4 - 3w - z). \tag{3}$$

It is easy to prove that (B)$\subset$(D) and hence that (C)$\subset$(D); in fact, every convergent product of inner functions is inner, and if each factor is good so is the product. For $N \geqq 2$, (C)$\neq$(D): Let Φ be any inner function in one variable, let α be a complex number, $0 < |\alpha| < 1$, and put

$$g(z, w) = (z\Phi(w) - \alpha)/(1 - \bar{\alpha}z\Phi(w)). \tag{4}$$

That g is good follows quite easily via Jensen's formula; one can choose Φ so that the zero-set of g is not a countable union of algebraic varieties, and then $g \notin$ (C).

A continuous function in U^N is called N-harmonic if it is harmonic in each of the variables $z_1, \cdots, z_N$. If $\Phi \in L^1(T^N)$ and if μ is a measure on T^N, the Poisson integrals $P[\Phi]$ and $P[d\mu]$ are N-harmonic in U^N [4; pp. 303, 315]. If $f \in H^\infty(U^N)$ and $f \not\equiv 0$, put $f_r(z) = f(rz)$ $(0 \leqq r < 1, z \in T^N)$ and define

$$u[f] = \lim_{r \to 1} P[\log |f_r|]. \tag{5}$$

The limit exists (compare [4; pp. 321–322]) and is the *least N-harmonic majorant of* $\log |f|$.

For inner functions g, $u[g] \leqq 0$, and g is good if and only if $u[g] = 0$.

We let RP denote the class of all functions u which are *real parts* of holomorphic functions $u + iv$ in U^N. Every $u \in$ RP is clearly N-harmonic.

After these preliminaries we can state some results. The first two are quite easy:

THEOREM 1. *If $f \in H^\infty(U^N)$, g is a good inner function in U^N, and f/g is holomorphic in U^N, then $f/g \in H^\infty(U^N)$.*

THEOREM 2. *Suppose $f \in H^\infty(U^N)$, $f \not\equiv 0$.*

(a) *If $u[f]$ is in* RP *then there is a unique (up to multiplication by constants) good inner function g which has the same zeros as f.*

(b) *If $u[f]$ is not in* RP *then no good inner function has the same zeros as f.*

THEOREM 3. *Suppose $\Phi \in L^1(T^N)$, $\Phi > 0$, and Φ is lower semicontinuous. Then there is a singular positive measure σ on T^N such that the Poisson integral $P[\Phi - d\sigma]$ is in* RP.

PROOF. $\Phi = \sum \Phi_k$, where each Φ_k is a positive trigonometric polynomial on T^N. Denote the Fourier coefficients of Φ_k by $\hat{\Phi}_k(n_1, \cdots, n_N)$. Fix k. If p_k is a positive integer, there is a positive singular measure σ_k on T^N such that

$$(6) \qquad \hat{\sigma}_k(n_1, \cdots, n_N) = \sum_{j=-\infty}^{\infty} \hat{\Phi}_k(n_1 + jp_k, \cdots, n_N + jp_k).$$

If p_k is large enough, $P[\Phi_k - d\sigma_k]$ is in RP. Put $\sigma = \sum \sigma_k$.

THEOREM 4. *If ψ is a bounded positive lower semicontinuous function on T^N then there exists $f \in H^\infty(U^N)$ with $|f^*| = \psi$ a.e.*

PROOF. The hypothesis implies that ψ has a positive lower bound. Assume $\psi > 1$, without loss of generality. Apply Theorem 3 to $\Phi = \log \psi$, put $u = P[\Phi - d\sigma]$, and define $f = \exp(u + iv)$. Note that $P[d\sigma]$ has radial limit 0 a.e. since σ is singular [4; p. 313].

The assumed lower semicontinuity of ψ is of course not a necessary condition for the existence of an $f \in H^\infty(U^N)$ with $|f^*| = \psi$ a.e. Nevertheless it is not an entirely unnatural hypothesis since a certain amount of lower semicontinuity is forced on $|f^*|$: If $f \in H^\infty(U^N)$, $z \in T^N$, and $f_z(\lambda) = f(\lambda z)$, then $f_z \in H^\infty(U)$, so that

$$(7) \qquad \operatorname*{ess\,sup}_{|\lambda|=1} |f^*(\lambda z)| = \sup_{|\lambda|<1} |f(\lambda z)| \qquad (z \in T^N).$$

The right side of (7) is clearly a lower semicontinuous function on T^N, hence so is the left.

THEOREM 5. *Suppose $f \in H^\infty(U^N)$, $f \neq 0$, ψ is an upper semicontinuous function on T^N, and $|f^*| = \psi$ a.e. Then there is an inner function g with the same zeros as f.*

PROOF. By Theorem 3, applied to $\Phi = -\log \psi$, there is a positive singular measure σ on T^N such that

$$(8) \qquad u = P[\log \psi + d\sigma]$$

is in RP. If $h = \exp(u + iv)$, then $|h^*| = |f^*|$ a.e., and

$$(9) \qquad \log |f| \leqq P[\log |f^*|] = P[\log \psi] \leqq u = \log |h|.$$

Put $g = f/h$.

The theorem applies, in particular, to any $f \in H^\infty(U^N)$ which has a continuous extension to the closure of U^N. But it may be impossible to choose g so that f/g is bounded in U^N, even if f is a polynomial! To state this more precisely, let V^N be the set of all $(z_1, \cdots, z_N)$ with $|z_j| > 1$ for $j = 1, \cdots N$.

THEOREM 6. *Let f be a polynomial in $z_1, \cdots, z_N$, $f \not\equiv 0$.*

(a) *If f has no zero in V^N then there is a rational inner function g such that f/g is a polynomial with no zero in U^N.*

(b) *If f is irreducible and if f has zeros in both U^N and V^N then f/g is unbounded in U^N for every inner function g which has the same zeros as f in U^N.*

PROOF. (a) is trivial (see [2; p. 991]). To prove (b), assume f/g is bounded in U^N, so $|g| \geqq c|f|$ for some $c>0$. Hence, for almost all $z \in T^N$, $g(\lambda z)$ is a finite Blaschke product. This implies (via Theorem 2.1 of [2]) that g is rational. Since f is irreducible it follows that every zero of f (in C^N) is a zero of g. But rational inner functions have no zeros in V^N.

Note that (b) may hold even if f has no zero on T^N: $f(z, w) = z+2w$.

The next result should be compared with Theorem A of [3].

THEOREM 7. *There exists $f \in H^\infty(U^2)$, $f \not\equiv 0$, with the following property: For no holomorphic function h in U^2 is fh an inner function.*

PROOF. Let A be an open set on the unit circle whose complement is totally disconnected and has positive measure, let E be the set of all $(z, w) \in T^2$ such that $z/w \in A$. By Theorem 4 there exists $F \in H^\infty(U^2)$ with $|F^*| = 1$ a.e. on E, $|F^*| = \frac{1}{2}$ a.e. off E. A generalization of a theorem of Frostman [1; pp. 111–113] shows that there exist arbitrarily small α such that

$$u[F - \alpha] = P[\log |F^* - \alpha|]. \tag{10}$$

Put $f = F - \alpha$, and suppose (to get a contradiction) that fh is inner for some holomorphic h. By (10), $u[f]$ is bounded below, so $u[h]$ is bounded above, hence $h \in H^\infty(U^N)$. But $|h^*|$ is close to 1 a.e. on E and $|h^*|$ is close to 2 a.e. on the rest of T^N. This violates the lower semicontinuity property discussed after Theorem 4.

A more detailed discussion of these results, including complete proofs and extensions to other H^p-spaces, will be published elsewhere.

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UNIVERSITY OF WISCONSIN AND
UNIVERSITY OF CALIFORNIA, SAN DIEGO