AN EXAMPLE IN THE CALCULUS OF FOURIER TRANSFORMS

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0. The functions which operate on Fourier or Fourier-Stieltjes transforms have been investigated by Helson, Kahane, Katznelson, and Rudin, especially in [1]. In this note we give an example of a positive measure on the Cantor group D_2, whose Fourier-Stieltjes transform has range in $[0, 1]$, and on which the continuous functions operating must have a high degree of analyticity. Our method of expanding this function is based on Bernstein polynomials and is quite different from that of [1].

1. Let D_2 be the complete direct sum $Z_2 \oplus Z_2 \oplus Z_2 \oplus \cdots$, e_n the unit mass at 0 in the nth factor, m_n the uniform $(1/2, 1/2)$ mass in the same group. For a dense sequence $\{a_n\} \subseteq [0, 1]$ we form the infinite product measure

$$\mu = \prod_1^\infty \{a_n e_n + (1 - a_n) m_n\}.$$

Denote by W the set of complex numbers $\{|z| < 1\} - \{-1 < z \leqq 0\}$.

THEOREM. *If f is continuous in $[0, 1]$ and $f \circ \hat{\mu}$ is a Fourier-Stieltjes transform on $\hat{D}_2$, then f can be extended to a function bounded and analytic in W.*

The proof is based on certain measures σ on the N-fold sum $Z_2 \oplus Z_2 \oplus \cdots \oplus Z_2$, in which each element is an N-tuple $(x_1, x_2, \cdots, x_N)$ $(x_i = 0, 1, 1 \leqq i \leqq N)$. Say that σ is *special* if it is invariant with respect to permutations of the coordinates $x_1, \cdots, x_N$. A special measure is a linear combination of the measures σ_j, $0 \leqq j \leqq N$, described as follows: σ_j assigns mass 1 to every element x for which $\sum_{i=1}^N x_k = j$.

For any special measure σ there are defined numbers $b_0, \cdots, b_N$: b_k is the value of $\hat{\sigma}$ on the character

$$x \to (-1)^{\sum_{i=1}^k x_i}.$$

LEMMA. *For a special measure σ, set*

$$B(x) = \sum_0^N b_k \binom{N}{k} x^k (1 - x)^{N-k}.$$

Then

$$B(x) = \sum_0^N c_k(1 - 2x)^k, \quad \textit{with} \quad \sum_0^N |c_k| \leq \|\sigma\|.$$

PROOF. Because the measures $\sigma_0, \cdots, \sigma_N$ are mutually singular it is enough to verify the estimate for each σ_j. The number b_k is the coefficient of s^j in $(1 - s)^k(1 + s)^{N-k}$; we write this as $(j!)^{-1}\partial^j/\partial s^j\,[(1-s)^k(1+s)^{N-k}]$ (the derivative is ultimately evaluated at $s=0$). Then

$$\begin{aligned} B(x) &= (j!)^{-1}\frac{\partial^j}{\partial s^j}\left[\sum_{k=0}^N \binom{N}{k} x^k(1 - x)^{N-k}(1 - s)^k(1 + s)^{N-k}\right] \\ &= (j!)^{-1}\frac{\partial^j}{\partial s^j}(1 + s - 2xs)^N \\ &= \binom{N}{j}(1 - 2x)^j, \qquad \text{at } s = 0. \end{aligned}$$

Since

$$\|\sigma_j\| = \binom{N}{j},$$

the lemma is proved.

To prove the theorem, we provide D_2 with coordinates $(x_1, x_2, \cdots, x_n, \cdots)$ and let ξ_n be the character $x \rightarrow (-1)^{x_n}$ $(1 \leq n < \infty)$. Now if $n_1 < n_2 < \cdots < n_N$ and $\epsilon_i = 0, 1$ $(1 \leq i \leq N)$

$$\hat{\mu}(\epsilon_1\xi_{n_1} + \cdots + \epsilon_N\xi_{n_N}) = \prod_{i=1}^N \{1 + \epsilon_i(a_{n_i} - 1)\},$$

and

$$f \circ \hat{\mu}(\epsilon_1\xi_{n_1} + \cdots + \epsilon_N\xi_{n_N}) = f(\textstyle\prod).$$

Since $\{a_n\}$ is dense in $[0, 1]$, we see by choosing the indices $n_1, \cdots, n_N$ carefully that, for every N and every $b \in [0, 1]$, the function g_b defined on the dual of Z_2^N by the formula

$$g_b(\epsilon_1, \cdots, \epsilon_N) = f\left(b \sum_1^N \epsilon_i\right)$$

is the transform of a *special* measure on Z_2^N with norm at most $\|f(\mu)\|$.

For each $r>0$ define $\phi_r(u)=f(e^{-ru})$, $0\leqq u\leqq 1$. Then ϕ_r is the uniform limit of the Bernstein polynomials

$$B_N(u) = \sum_0^N \binom{N}{k} \phi_r(k/N)u^k(1-u)^{N-k},$$

$$(0 \leqq u \leqq 1, \quad N = 1, 2, 3, \cdots)$$

$$= \sum_0^N \binom{N}{k} f(e^{-rk/N})u^k(1-u)^{N-k},$$

Widder [2].

By the lemma, and the subsequent remarks,

$$B_N(u) = \sum_0^N c_{k,N}(1-2u)^k \quad \text{with} \quad \sum_0^N |c_{k,N}| \leqq \|f(\mu)\|.$$

Clearly the B_N's form a normal family in $\{|u-1/2|<1/2\}$, so that ϕ_r can be extended to a function analytic in this open set and bounded by $\|f(\mu)\|$. But this just means that f can be extended to an analytic function of (the principal branch) of log, in the set where log has real part between e^{-r} and 1. As $r\to\infty$ the theorem is proved.

References

1. H. Helson, J.-P. Kahane, Y. Katznelson, and W. Rudin, *The functions which operate on Fourier transforms*, Acta Math. **102** (1959), 135–157.

2. D. V. Widder, *The Laplace transform*, Princeton Univ. Press, Princeton, N. J., 1942.

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