CONSTRUCTIVE TRANSFINITE NUMBER CLASSES

BY WAYNE RICHTER[1]

Communicated by D. Scott, September 13, 1966

1. Introduction. The notion of an ordinal system *restricted productive* with respect to given sets was introduced in [9] and used to define constructive finite number classes. It was shown that both the forms of the sets of notations for the finite number classes and the ordinals obtained are the same as of the sets O, O^O, O^{O^O}, $\cdots$, and the ordinals $\omega_1 < \omega_1^O < \omega_1^{O^O} < \cdots$, respectively. In this article these results are extended to constructive transfinite number classes. We present an ordinal system $(F, |\ |)$ which, in terms of our analogy with the classical ordinals, provides notations for the ordinals less than the first "constructively inaccessible" ordinal. Knowledge of the theory of constructive ordinals suggests that this should lead to a natural class of ordinals of some independent interest. This is born out by the characterization of the ordinals of $(F, |\ |)$ given below. $\boldsymbol{E}_1$ is the type-2 representing functional of the predicate $\lambda\alpha.(\forall\beta)(\exists x)[\alpha(\bar{\beta}(x))=0]$ introduced by Tugué [12] (see also Kleene [4]). Let $\omega_1^{E_1}$ be the smallest ordinal which is not the order type of any well-ordering recursive in $\boldsymbol{E}_1$. Our principal result is that the system $(F, |\ |)$ provides notations for exactly the ordinals less than ω^{E_1}, and the sets of notations for the number classes form an $\boldsymbol{E}_1$-hierarchy.

Kreider-Rogers [5] discussed three systems of notations, each of which regarded internally provides an analogue with the ordinals less than the first inaccessible, but it is not clear that any of these systems gives a natural class of ordinals. It is clear from Theorem 2 below that $(F, |\ |)$ provides notations for at least all of the ordinals of the systems in [5], but the question of equivalence remains open.

Related results are obtained about initial ordinals and hierarchies independent of systems of notations. Proofs will appear elsewhere. Notation used is similar to that of [9].

The author is indebted to R. O. Gandy for his communication of a conjecture which led to Theorem 2.

[1] Research of the author was begun while he held a National Science Foundation Predoctoral fellowship. The research was continued at Dartmouth College in the summer of 1964 supported by National Science Foundation Grant G23805, and at the University of California, Berkeley, during the academic year 1964–65 while the author held a faculty fellowship from Rutgers, The State University.

2. **The system** $(F, |\ |)$. The ordinal system $(F, |\ |)$ is similar to the system C of [5] except the order-preserving requirement at limit ordinals is omitted. Our presentation parallels the formulation of C given in Putnam [7]. N_ν is the set of notations for the ordinal ν. If for some ν, $x \in N_\nu$, we let $|x| = (\mu\xi)[x \in N_\xi]$. $C_\nu = \bigcup\{N_\xi : \xi < \nu\}$. i is a Gödel number of the identity function. n is an *index* in C_ν if $3^n 5^t \in C_\nu$ for some t. $F = \bigcup N_\nu$.

Case 1. $\nu = 0$. Then $N_\nu = \{1\}$.

Case 2. $\nu = \xi + 1$, where N_ξ is already defined. Then $N_\nu = \{2^x : x \in N_\xi\}$.

Case 3. ν is a limit ordinal such that N_γ is already defined for all $\gamma < \nu$, and there exists an ordinal $\xi < \nu$ such that $3^a 5^i \in N_\xi$ for some a and a partial recursive function f such that $C_\xi \subseteq \delta f$ and $f(C_\xi) \subseteq C_\nu$ and $\nu = \text{lub}\{|f(t)| : t \in C_\xi\}$. Then N_ν is taken to be the set of all numbers $3^a 5^n$ such that $3^a 5^i \in N_\xi$, where ξ is any ordinal with the above property, and $C_\xi \subseteq \delta\{n\}$ and $\{n\}(C_\xi) \subseteq C_\nu$ and $\nu = \text{lub}\{|\{n\}(t)| : t \in C_\xi\}$.

Case 4. ν is a limit ordinal such that N_γ is already defined for all $\gamma < \nu$, and Case 3 does not hold, but there is a number $a \in C_\nu$ which is not an index in C_ν. Let $\xi < \nu$ be the smallest ordinal such that N_ξ contains a nonindex in C_ν. Then

$$N_\nu = \{3^a 5^n : a \in N_\xi \wedge C_\nu \subseteq \delta\{n\} \wedge \{n\}(C_\nu) \subseteq C_\nu \wedge \nu = \text{lub}\{|\{n\}(t)| : t \in C_\nu\}\}.$$

$x \in N_\nu$ only as required by Cases 1–4.

It is easy to verify that if $\nu \neq \xi$ then $N_\nu \cap N_\xi = \emptyset$. The smallest ordinal for which there is no notation in F is denoted by $|F|$. For $x \in F$, $F_{|x|} = C_{|3^x 5^i|}$ is the set of notations for the $|x|$th (cumulative) number class of $(F, |\ |)$($|x|+1$st if $|x| < \omega$). We also let $F_{|F|} = F$. For $\nu \leqq |F|$, let $F_\nu^* = \bigcup\{F_\xi : \xi < \nu\}$. The smallest ordinal for which there is no notation in F_ν^* is denoted by $|F_\nu^*|$.

It is not difficult to show that the number classes (and F itself) are (uniformly) restricted productive with respect to smaller number classes. Then using the techniques of [9], the recursion theorem, and a proof by transfinite induction, we obtain:

THEOREM 1. *For* $1 \leqq \nu \leqq \xi < |F|$,

(1) $F_\nu \cong O^{F_\nu^*}$,

(2) $|F_\nu| = \omega_1^{F_\nu^*}$,

(3) $F_\nu^* \leqq_1 F_\xi^* \leqq_1 F$.

In particular, $F_{\nu+1} \cong O^{F_\nu}$ and $|F_{\nu+1}| = \omega_1^{F_\nu}$. The proof of Theorem 1 gives the stronger result, used in the applications, that the isomorphisms in (1) and the reducibilities in (3) may be found effectively from a notation for ν.

THEOREM 2. (1) *A set A is recursive in* E_1 *iff for some* $\nu < |F|$, *A is recursive in* $F\nu$,

(2) $|F| = \omega_1^{E_1}$,

(3) $F \cong O^{E_1}$,

(4) *F is a complete set of the form* $\{x: (\exists\alpha)^{E_1} P(x, \alpha)\}$, *where the predicate P is recursive in* E_1 *and the subscript* E_1 *means the range of the quantifier is restricted to number-theoretic functions recursive in* E_1,

(5) *F is not recursive in* E_1.

(1) says that the number classes of $(F, |\ |)$ form an E_1-hierarchy. The problem of finding an E_1-hierarchy was raised by Shoenfield [10]. Shoenfield [11] has since announced a method for constructing a hierarchy for an arbitrary type-2 functional in which E (defined in Kleene [3]) is recursive. Observe that Theorem 2 is an analogue of familiar properties of sets of notations for the constructive ordinals. Specifically, Theorem 2 remains true if F and E_1 are replaced throughout by O and E respectively.

It is clear that $(F, |\ |)$ is a D-system with arithmetic ϕ (defined in Putnam [8]). Hence from [8] it follows that $F \in \Delta_2^1$. This implies by (5) the result (previously obtained by Shoenfield [10] and Gandy [2]) that the functions recursive in E_1 are a proper subclass of Δ_2^1.[2]

3. **Initial ordinals and hierarchies independent of systems of notations.** The following definition of initial number is due to Gandy [2].

DEFINITION. (1) An ordinal ν is called *regular* if for some set A, $\nu = \omega_1^A$. $\nu > \omega$ is *initial* if it is regular or is a limit of regular ordinals. Let $\omega_0 = \omega$, and for $\nu > 0$ let ω_ν be the νth initial ordinal greater than ω.

(2) For $1 \leq \nu < \omega_1^{E_1}$ let $\omega_\nu^\# = |F_\nu^*|$ ($\omega_{\nu-1}^\#$ if $\nu < \omega$). $\omega_\nu^\#$ is called the νth ($\nu+1$st if $\nu < \omega$) *initial ordinal of* $(F, |\ |)$.

THEOREM 3. *If* $\nu < \omega_1^{E_1}$, *then* $\omega_\nu^\# = \omega_\nu$.

Thus the initial numbers coincide with the initial numbers of $(F, |\ |)$ for $\nu < \omega_1^{E_1}$.

We do not know at present if there exist sets A of nonnegative integers and limit ordinals ν such that $\omega_\nu = \omega_1^A$. However, from the basis theorem for Δ_2^1 sets (Addison [1]) it follows that if such sets A exist then there also exist such sets belonging to Δ_2^1. On the other hand we have:

---

[2] Since the well-ordering functional W ($W(\alpha) = 0$ or 1, depending on whether or not α is a well-ordering) is equivalent to E_1, it also follows from (5) that there are D-systems whose sets of notations are not recursive in W. Thus, D-systems are more powerful than was anticipated in [6].

THEOREM 4. *A limit ω_ν of an increasing sequence of regular ordinals is not itself a regular ordinal for $\nu < \omega_1^{E_1}$. And if ω_1^A is the limit of an increasing sequence of regular ordinals, then A is not recursive in $\boldsymbol{E}_1$.*

A hierarchy of hyperdegrees may be defined independently of systems of notations by the following procedure. Let $\boldsymbol{h}_1$ be the hyperdegree of hyperarithmetic sets. If $\boldsymbol{h}_\nu$ has been defined, let $\boldsymbol{h}_{\nu+1}$ be the hyperjump of $\boldsymbol{h}_\nu$. If ν is a limit ordinal and $\boldsymbol{h}_\xi$ has been defined for all $\xi < \nu$, let $\boldsymbol{h}_\nu$ be the least upper bound of $\{\boldsymbol{h}_\xi : \xi < \nu\}$, provided the least upper bound exists. We do not know at what limit ordinal τ this hierarchy terminates.

THEOREM 5. *For $1 \leqq \nu < \omega_1^{E_1}$, $F_\nu^* \in \boldsymbol{h}_\nu$.*

Thus $\tau \geqq \omega_1^{E_1}$. We conjecture that $\tau = \omega_1^{E_1}$ and that $\omega_1^{E_1}$ is regular.

REMARK. We note in connection with Theorems 4 and 5 that for any A: (1) $\omega_1^{E_1} < \omega_1^{A_1}$ iff F is hyperarithmetic in A, and (2) if the hyperdegree of A is an upper bound of $\{\boldsymbol{h}_\xi : \xi < \omega_1^{E_1}\}$ then $F \leqq_1 O^A$. Thus the hyperdegree of F is "almost" the lub of $\{\boldsymbol{h}_\xi : \xi < \omega_1^{E_1}\}$.

The hyperdegrees $\boldsymbol{h}_\xi$, $\xi < \omega_1^{E_1}$ may be characterized in terms of Π_1^1 singletons as follows.

THEOREM 6. *For every set A recursive in $\boldsymbol{E}_1$,*

$$A \in \bigcup\{\boldsymbol{h}_\xi : \xi < \omega_1^{E_1}\} \Leftrightarrow \{A\} \in \Pi_1^1.$$

Suzuki [13] showed that

$$(\forall A)\Delta_2^1(\exists B)\Delta_2^1[A \leqq_1 B \wedge \{B\} \in \Pi_1^1].$$

From Theorems 2 and 6 we have:

COROLLARY 7. $(\forall A)_{E_1}(\exists B)_{E_1}[A \leqq_1 B \wedge \{B\} \in \Pi_1^1]$.

4. Extensions of $(F, |\ |)$. $(F, |\ |)$ can be extended by adding notations for higher order inaccessibles along the lines described in [5, p. 368]. Specifically, let $\mathcal{C}_1 = (F, |\ |)$, and for $1 < n < \omega$, let $\mathcal{C}_n = (C_n, |\ |_n)$ be the extension of $\mathcal{C}_1$ obtained by adding notations for the points of nth order difficulty. For any type-2 functional $\boldsymbol{F}$ let $d(\boldsymbol{F})$ be a type-2 functional equivalent to the representing functional of the predicate $[\lambda x\alpha . \{x\}(\alpha, \boldsymbol{F})$ is defined$]$. For $1 < n < \omega$, let $\boldsymbol{E}_{n+1} = d(\boldsymbol{E}_n)$. Then Theorem 2 remains true if $(F, |\ |)$ is replaced by $\mathcal{C}_n$ and $\boldsymbol{E}_1$ is replaced by E_n.[3] This procedure can presumably be extended into the transfinite. The general situation is under investigation.

Added in proof. Saul A. Kripke has informed the author that he has independently shown that $\tau \geqq \omega_1^{E_1}$, and also that τ is a Δ_2^1 ordinal.

References

1. J. W. Addison, *Hierarchies and the axiom of constructibility*, 2nd ed., Summer Institute for Symbolic Logic, Cornell University, Ithaca, N. Y., 1957, pp. 355–362, Institute for Defense Analyses, Princeton, New Jersey, 1960.

2. R. O. Gandy, *General recursive functionals of finite type and hierarchies of functions* (mimeographed), 1962.

3. S. C. Kleene, *Recursive functionals and quantifiers of finite types*. I, Trans. Amer. Math. Soc. 91 (1959), 1–52.

4. ———, *Recursive functionals and quantifiers of finite types*. II, Trans. Amer. Math. Soc. 108 (1963), 106–142.

5. D. L. Kreider and H. Rogers, Jr., *Constructive versions of ordinal number classes*, Trans. Amer. Math. Soc. 100 (1961), 325–369.

6. G. Kreisel, *Review of "On hierarchies and systems of notation,"* Math. Reviews 28 (1964), 224.

7. H. Putnam, *Uniqueness ordinals in higher constructive number classes*, *Essays on the foundations of mathematics*, The Hebrew University, Magnes Press, Jerusalem, 1961, pp. 190–206.

8. ———, *On hierarchies and systems of notations*, Proc. Amer. Math. Soc. 15 (1964), 44–50.

9. W. Richter, *Extensions of the constructive ordinals*, J. Symbolic Logic 30 (1965), 193–211.

10. J. R. Shoenfield, *The form of the negation of a predicate*, *Recursive function theory*, Proc. Sympos. Pure Math., Vol. 5, pp. 131–134, Amer. Math. Soc., Providence, R. I., 1963.

11. ———, *A hierarchy for objects of type* 2, Abstract 65T-173, Notices Amer. Math. Soc. 12 (1965), 369–370.

12. T. Tugué, *Predicates recursive in a type-*2 *object and Kleene hierarchies*, Comment. Math. Univ. St. Paul (Tokyo) 8 (1960), 97–117.

13. Y. Suzuki, *A complete classification of the Δ_2^1-functions*, Bull. Amer. Math. Soc. 70 (1964), 246–253.

Rutgers, The State University

[3] Each $\mathcal{C}_n$ is a D-system. Thus the first ordinal for which there is no notation in D-systems is $\geqq \lim \omega_1^{E_n}$. In correspondence, R. O. Gandy states that in fact this ordinal equals $\lim \omega_1^{E_n}$.