A NONABELIAN TWO-DIMENSIONAL COHOMOLOGY FOR ASSOCIATIVE ALGEBRAS[1]

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In a previous paper [4], a nonabelian cohomology theory for associative algebras was proposed for dimensions 0 and 1. The purpose of this paper is to extend this theory to dimension 2. The methods used are closely analogous to those employed in the corresponding theory for groups (cf. [1],[2]). Throughout this paper all algebras will be associative algebras over some fixed commutative ring Λ with identity. The term *homomorphism* without qualification will mean homomorphism of Λ-algebras.

1. **One-cochains and one-cocycles.** We consider algebras B and M, and homomorphisms $\rho: B\to M$, $\Phi: M\to\mathfrak{M}(B)$, where $\mathfrak{M}(B)$ denotes the algebra of bimultiplications of B (cf. [4],[5]). The system $\mathfrak{B}=(B, M; \rho, \Phi)$ is said to define the structure of an *M-crossed module* on B if the following conditions are satisfied:

(i) the image of Φ is *permutable* on B;

(ii) $\rho(\Phi(m)b)=m\rho(b)$, $\rho(b\Phi(m))=\rho(b)m$, for all $b\in B$, $m\in M$;

(iii) the composite $\Phi\rho: B\to\mathfrak{M}(B)$ maps each element of B onto the inner bimultiplication which it defines.

We recall that a subset S of $\mathfrak{M}(B)$ is *permutable* on B if $(\xi b)\eta=\xi(b\eta)$, for all $\xi, \eta\in S$, $b\in B$ (cf. [4],[5]). The M-crossed module B is an M-bimodule under the action defined by Φ, and the condition (ii) shows that ρ is a homomorphism of M-bimodules. Moreover $\rho(B)$ is an ideal in M and the quotient algebra $L=M/\rho B$ has a canonical map $\Psi: L\to\mathfrak{M}(Z)$, deduced from Φ, where $Z=\operatorname{Ker}(\rho)$ is called the *center* of the crossed module $\mathfrak{B}$. M and L will be called the *operator* and *outer operator* algebras of $\mathfrak{B}$ respectively.

Given $\mathfrak{B}=(B, M; \rho, \Phi)$, we define a 1-*cochain* from an algebra A to $\mathfrak{B}$ to be a pair (p, ϕ) of maps $p: A\to B$, $\phi: A\to M$ and denote the set of these 1-cochains by $\mathfrak{C}^1(A, \mathfrak{B})$. $\mathfrak{Z}^1(A, \mathfrak{B})$ is the subset consisting of the 1-*cocycles*, namely those pairs (p, ϕ) for which

$$\begin{aligned}&\phi \text{ is a homomorphism;}\\ (1)\quad &p \text{ is a homomorphism of } \Lambda\text{-modules;}\\ &p(a_1a_2) = p(a_1)p(a_2) + p(a_1)\phi(a_2) + \phi(a_1)p(a_2), \qquad a_1, a_2 \in A.\end{aligned}$$

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The composition $(q, \psi)+(p, \phi)$ in $\mathcal{C}^1(A, \mathcal{B})$ is defined only if $\psi=\rho p+\phi$, in which case

$$(q, \psi) + (p, \phi) = (q + p, \phi). \tag{2}$$

$\mathcal{C}^1(A, \mathcal{B})$ together with this composition form a *groupoid*. From (1) we see that $\mathcal{Z}^1(A, \mathcal{B})$ forms a *subgroupoid* of $\mathcal{C}^1(A, \mathcal{B})$.

2. **Two-cochains and two-cocycles.** From the algebra A, we define $F(A)$ to be the free Λ-module on the generators $\{\bar{a} \mid a \in A\}$. The map $\bar{a} \to a$ extends to an epimorphism $F(A) \to A$ of Λ-modules, with kernel $N(A)$. Hence $\sum_i \lambda_i \bar{a}_i \in N(A)$ for $\lambda_i \in \Lambda$, $a_i \in A$, if and only if $\sum_i \lambda_i a_i = 0$.

A *2-cochain* from A to $\mathcal{B}$ is defined to be a triple $(\gamma_1, \gamma_2, \phi)$, of maps

$$\gamma_1 \colon A \times A \to B, \qquad \gamma_2 \colon N(A) \to B, \qquad \phi \colon A \to M.$$

A *2-cocycle* is a 2-cochain which satisfies the conditions:

(I) $\phi(a_1)\gamma_1(a_2, a_3) - \gamma_1(a_1a_2, a_3) + \gamma_1(a_1, a_2a_3) - \gamma_1(a_1, a_2)\phi(a_3) = 0$;

(II) $\sum_i \lambda_i \gamma_1(a, a_i) = \gamma_2(\sum_i \lambda_i \overline{aa_i}) - \phi(a)\gamma_2(\sum_i \lambda_i \bar{a}_i)$;

(III) $\sum_i \lambda_i \gamma_1(a_i, a) = \gamma_2(\sum_i \lambda_i \overline{a_i a}) - \gamma_2(\sum_i \lambda_i \bar{a}_i)\phi(a)$;

(IV) γ_2 is a homomorphism of Λ-modules;

(V) $\rho\gamma_2(\sum_i \lambda_i \bar{a}_i) = \sum_i \lambda_i \phi(a_i)$;

(VI) $\rho\gamma_1(a_1, a_2) = \phi(a_1a_2) - \phi(a_1)\phi(a_2)$;

for $\lambda_i \in \Lambda$, $a, a_1, a_2, a_3, a_i \in A$, $\sum_i \lambda_i \bar{a}_i \in N(A)$.

$C^2(A, \mathcal{B})$ and $Z^2 = Z^2(A, \mathcal{B})$ will denote respectively the sets of 2-cochains and 2-cocycles. Note that $(0, 0, \phi) \in Z^2$ iff $\phi \colon A \to M$ is a homomorphism. These special 2-cocycles form a privileged subset $I^2 = I^2(A, \mathcal{B})$ of Z^2, the elements of which are called *neutral*. Sometimes a special such ϕ is chosen corresponding to a *base cocycle* in Z^2; if ϕ is the zero morphism, then one speaks of the *zero-cocycle*.

Every extension $E \equiv 0 \to B \to^{\iota} E \to^{\epsilon} A \to 0$ of algebras gives rise to a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \to B & \xrightarrow{\iota} & E & \xrightarrow{\epsilon} & A & \to 0 \\ \| & & \downarrow \Theta & & \downarrow \theta & \\ B & \to & \mathfrak{M}(B) & \to & \mathcal{P}(B) & \to 0 \end{array} \tag{3}$$

where $\mathcal{P}(B)$ denotes the algebra of *outer bimultiplications* (cf. [**4**], [**5**]), and the homomorphism $\Theta \colon E \to \mathfrak{M}(B)$ is given by the multiplication in E. If further, B is an M-crossed module, and we have a homomorphism $\mu: E \to M$ such that $\Phi\mu = \Theta$, then the couple $(\boldsymbol{E}, \mu)$ will be called an *M-extension*. We then have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & 0 & \to & B & \xrightarrow{\iota} & E & \xrightarrow{\epsilon} & A & \to & 0 \\ & & & & \| & & \downarrow \mu & & \downarrow \nu & & \\ 0 & \to & Z & \to & B & \xrightarrow{\rho} & M & \to & L & \to & 0 \end{array} \tag{4}$$

and the homomorphism ν is called the *crest* of the M-extension. Two M-extensions $(\boldsymbol{E}, \mu)$, $(\boldsymbol{E}', \mu')$ are equivalent if there is a morphism $\boldsymbol{E} \to \boldsymbol{E}'$ of short exact sequences which extends the identity maps on B and A, and for which the composite homomorphism $E \to E' \to^{\mu'} M$ is precisely μ. We denote by $\operatorname{Ext}(A, \mathcal{B})$ the set of equivalence classes of M-extensions. The crest ν in (4) corresponds to a map

$$\kappa: \operatorname{Ext}(A, \mathcal{B}) \to \operatorname{Hom}(A, L). \tag{5}$$

$(\boldsymbol{E}, \mu)$ is called *inessential* if $\boldsymbol{E}$ is *cleft*, i.e., if there exists a homomorphism of algebras $t: A \to E$ such that $\epsilon t = 1_A$; it is called *trivial* if E is the direct sum algebra of B and A, and $\iota(b) = (b, 0)$, $\epsilon(b, a) = a$, and if $\mu = 0$.

3. **Cohomology classes.** It is now possible to define a map δ: $\mathcal{C}^1(A, \mathcal{B}) \to C^2(A, \mathcal{B})$. For $(p, \phi) \in \mathcal{C}^1(A, \mathcal{B})$, $\delta(p, \phi)$ is defined by the equation

$$\delta(p, \phi) = (\delta_1(p, \phi), \delta_2(p, \phi), \phi),$$

where, for $a_1, a_2 \in A$, $\sum_i \lambda_i \bar{a}_i \in N(A)$,

$$\delta_1(p, \phi)(a_1, a_2) = p(a_1 a_2) - p(a_1)p(a_2) - p(a_1)\phi(a_2) - \phi(a_1)p(a_2);$$

$$\delta_2(p, \phi)\left(\sum_i \lambda_i \bar{a}_i\right) = \sum_i \lambda_i p(a_i). \tag{6}$$

For $(q, \rho p + \phi)$, $(p, \phi) \in \mathcal{C}^1(A, \mathcal{B})$, we obtain from (2) and (6)

$$\begin{aligned} \delta_1\{(q, \rho p + \phi) + (p, \phi)\} &= \delta_1(q, \rho p + \phi) + \delta_1(p, \phi), \\ \delta_2\{(q, \rho p + \phi) + (p, \phi)\} &= \delta_2(q, \rho p + \phi) + \delta_2(p, \phi). \end{aligned} \tag{7}$$

Observe from (6) that $\delta(p, \phi) = (0, 0, \phi)$ if and only if the last two conditions of (1) are satisfied. Hence, for ϕ a homomorphism, $\delta(p, \phi) = (0, 0, \phi)$ iff $(p, \phi) \in \mathcal{Z}^1(A, \mathcal{B})$.

We now define an action Δ (on the left) of the groupoid $\mathcal{C}^1(A, \mathcal{B})$ on the sets $C^2(A, \mathcal{B})$ and $Z^2(A, \mathcal{B})$. This is given by maps

$$\Delta: \mathcal{C}^1(A, \mathcal{B}) \times C^2(A, \mathcal{B}) \to C^2(A, \mathcal{B}),$$
$$\Delta: \mathcal{C}^1(A, \mathcal{B}) \times Z^2(A, \mathcal{B}) \to Z^2(A, \mathcal{B}).$$

For $(p, \phi) \in \mathcal{C}^1(A, \mathcal{B})$, $(\gamma_1, \gamma_2, \phi) \in C^2(A, \mathcal{B})$, $\Delta\{(p, \phi) \times (\gamma_1, \gamma_2, \phi)\}$ is defined to be $(\gamma_1', \gamma_2', \phi')$, given by the equations

(8) $$\phi' = \rho p + \phi, \qquad \gamma_1' = \delta_1(p, \phi) + \gamma_1, \qquad \gamma_2' = \delta_2(p, \phi) + \gamma_2.$$

We easily verify that if $(\gamma_1, \gamma_2, \phi) \in Z^2(A, \mathcal{B})$, then so does $(\gamma_1', \gamma_2', \phi')$. It follows from (7) that, under Δ, $\mathcal{C}^1(A, \mathcal{B})$ is a groupoid of left operators.

The *orbits* in $Z^2(A, \mathcal{B})$ under Δ are called the *thick 2-cohomology classes*, and the set of these classes is denoted by $\boldsymbol{H}^2(A, \mathcal{B})$. This is a set with preferred subset of *neutral classes* (i.e. containing a neutral cocycle) and with an eventual *base* or *zero-class* (containing the base or the zero 2-cocycle). There are also maps k and κ' falling into the following commutative diagram with canonical horizontal arrow:

(9)
$$\begin{array}{ccc} Z^2(A, \mathcal{B}) & \longrightarrow & \boldsymbol{H}^2(A, \mathcal{B}) \\ & {\scriptstyle k}\searrow \quad \swarrow{\scriptstyle \kappa'} & \\ & \mathrm{Hom}\,(A, L) & \end{array}$$

and such that $k(\gamma_1, \gamma_2, \phi) = \pi \circ \phi$ with $\pi: M \to L$ canonical.

We see that both sets $\mathrm{Ext}(A, \mathcal{B})$ and $\boldsymbol{H}^2(A, \mathcal{B})$ are provided with a privileged subset, an eventual base point and maps κ and κ' as in (5) and (9). One can also show that an abelian group $H^2_\nu(A, Z)$ acts in a simply transitive way on the fibres $\kappa^{-1}(\nu)$ and $\kappa'^{-1}(\nu)$ in $\mathrm{Ext}(A, \mathcal{B})$ and $\boldsymbol{H}^2(A, \mathcal{B})$. Those sets therefore have *spider* structures in the sense of [6]. Moreover:

THEOREM 1. *There exists a bijection* $\mathrm{Ext}(A, \mathcal{B}) \to \boldsymbol{H}^2(A, \mathcal{B})$ *which is an isomorphism in the category of spiders* [6] *and which maps the trivial M-extension onto the null class.*

The bijection is given by associating with each extension a factor system, which then corresponds to a 2-cocycle. An equivalence class of extensions will then correspond to an orbit of 2-cocycles.

4. **Functorial properties.** The functorial properties we desire are best described by the introduction of the category $\mathbf{C}$. The objects of $\mathbf{C}$ are the systems $\mathcal{B} = \{B, M; \rho, \Phi\}$ of §1. A morphism $\tilde{\sigma}: \{B_1, M_1; \rho_1, \Phi_1\} \to \{B_2, M_2; \rho_2, \Phi_2\}$ in $\mathbf{C}$ is a pair $\sigma: B_1 \to B_2$, $s: M_1 \to M_2$ of algebra homomorphisms for which $\rho_2 \circ \sigma = s \circ \rho_1$ and $\sigma(bm) = \sigma(b)s(m)$, $\sigma(mb) = s(m)\sigma(b)$, for all $m \in M_1$, $b \in B_1$. (We write bm for $b\Phi_1(m)$, etc.)

Every morphism $\tilde{\sigma}$ in $\boldsymbol{C}$ induces a morphism $\mathcal{C}^1(A, \mathfrak{B}_1)\rightarrow\mathcal{C}^1(A, \mathfrak{B}_2)$ of groupoids, which maps $\mathcal{Z}^1(A, \mathfrak{B}_1)$ into $\mathcal{Z}^1(A, \mathfrak{B}_2)$. Under this induced morphism, $(p,\phi)\in\mathcal{C}^1(A, \mathfrak{B}_1)$ is mapped onto $(\sigma p, s\phi)\in\mathcal{C}^1(A, \mathfrak{B}_2)$. Also, $\tilde{\sigma}$ induces a map $Z^2(A, \mathfrak{B}_1)\rightarrow Z^2(A, \mathfrak{B}_2)$ which sends $(\gamma_1, \gamma_2, \phi)$ $\in Z^2(A, \mathfrak{B}_1)$ into $(\sigma\gamma_1, \sigma\gamma_2, s\phi)$. Since

$$(10)\qquad \sigma\delta_1(p, \phi) = \delta_1(\sigma p, s\phi), \qquad \sigma\delta_2(p, \phi) = \delta_2(\sigma p, s\phi),$$

the images of two 2-cocycles in the same orbit of $Z^2(A, \mathfrak{B}_1)$ also lie in the same orbit of $Z^2(A, \mathfrak{B}_2)$. Hence $\tilde{\sigma}$ induces a map Σ: $\boldsymbol{H}^2(A, \mathfrak{B}_1)$ $\rightarrow\boldsymbol{H}^2(A, \mathfrak{B}_2)$.

THEOREM 2. *For a fixed algebra A, $\mathcal{C}^1$, $\mathcal{Z}^1$, Z^2 and $\boldsymbol{H}^2$ are covariant functors with domain $\boldsymbol{C}$. The codomain of $\mathcal{C}^1$ and $\mathcal{Z}^1$ is the category of groupoids, and the codomain of Z^2 and $\boldsymbol{H}^2$ is the category of (based) spiders.*

5. **Exact sequences.** A sequence of morphisms in $\boldsymbol{C}$:

$$(11)\qquad \mathfrak{B}_1 \overset{\tilde{\sigma}_1}{\rightarrow} \mathfrak{B}_2 \overset{\tilde{\sigma}_2}{\rightarrow} \mathfrak{B}_3, \qquad \mathfrak{B}_i = \{B_i, M_i, \rho_i, \Phi_i\},$$

gives a commutative diagram

$$(12)\qquad \begin{array}{ccccccccc} 0 & \rightarrow & B_1 & \overset{\sigma_1}{\rightarrow} & B_2 & \overset{\sigma_2}{\rightarrow} & B_3 & \rightarrow & 0 \\ & & \downarrow \rho_1 & & \downarrow \rho_2 & & \downarrow \rho_3 & & \\ & & M_1 & \overset{s_1}{\rightarrow} & M_2 & \overset{s_2}{\rightarrow} & M_3. & & \end{array}$$

The sequence (11) will be called a *short-exact sequence* if

(13)
- the top row in (12) is an exact sequence of algebras;
- s_1 is an isomorphism of algebras;
- s_2 is an epimorphism of algebras.

The sequence (11) induces the sequence

$$(14)\qquad \boldsymbol{H}^2(A, \mathfrak{B}_1) \overset{\Sigma_1}{\rightarrow} \boldsymbol{H}^2(A, \mathfrak{B}_2) \overset{\Sigma_2}{\rightarrow} \boldsymbol{H}^2(A, \mathfrak{B}_3).$$

THEOREM 3. *If* (11) *is short-exact, then* (14) *is exact.*

The exactness of (14) is in the sense that, an element of $\boldsymbol{H}^2(A, \mathfrak{B}_2)$ is in the image of Σ_1 if and only if its image under Σ_2 is neutral.

Henceforth we assume we have a short-exact sequence (11). The *fibres* of the map Σ_1 in (14) can then be described in terms of an action $\diamondsuit$ of $\mathcal{Z}^1(A, \mathfrak{B}_3)$ on $\boldsymbol{H}^2(A, \mathfrak{B}_1)$. Specifically, we take $(\gamma_1, \gamma_2, \phi)$ $\in Z^2(A, \mathfrak{B}_1)$ and $(p, \psi)\in\mathcal{Z}^1(A, \mathfrak{B}_3)$ such that $\psi = s_2s_1\phi$. Then lift (p, ψ)

to $(q, s_1\phi) \in \mathfrak{C}^1(A, \mathfrak{B}_2)$ taking $q: A \to B_2$ such that $\sigma_2 q = p$. Check that $\Delta\{(q, s_1\phi)\ x(\sigma_1\gamma_1, \sigma_1\gamma_2, s_1\phi)\} \in Z^2(A, \mathfrak{B}_2)$ is the image under $\tilde{\sigma}_1$ of a (necessarily unique) element $(\beta_1, \beta_2, \chi) \in Z^2(A, \mathfrak{B}_1)$, where

$$\chi = s_1^{-1}(\rho_2 q + s_1\phi).$$

Check that the class of (β_1, β_2, χ) is independent of the choice of q and define the action $\diamondsuit$ setting

$$(p, \psi) \diamondsuit [\gamma_1, \gamma_2, \phi] = [\beta_1, \beta_2, \chi]$$

where the brackets indicate the cohomology class of a cocycle. This is a groupoid action, defined iff $\psi = s_2 s_1 \phi$.

THEOREM 4. *If* (11) *is short-exact, then the fibres of the map* Σ_1 *in* (14) *are the orbits in* $\boldsymbol{H}^2$ $(A, \mathfrak{B}_1)$ *under the* $\diamondsuit$*-action of* $\mathcal{Z}^1$ $(A, \mathfrak{B}_3.)$

6. **The longer exact sequence.** Take $\mathfrak{B} = \{B, M; \rho, \Phi\}$ in $\boldsymbol{C}$, and take $\phi: A \to M$ to be a *fixed* homomorphism. Denote by $Z^1_\phi(A, \mathfrak{B})$ the subset of $\mathcal{Z}^1(A, \mathfrak{B})$ of those elements of the form (p, ϕ). $Z^1_\phi(A, \mathfrak{B})$ is then endowed with a base point $(0, \phi)$ and with a structure of polypus (cf. [**1**], [**4**]). In $\boldsymbol{H}^2(A, \mathfrak{B})$, the class of $(0, 0, \phi)$ will be called *ϕ-null*, and with this as base point, our based set will be denoted by $\boldsymbol{H}^2_\phi(A, \mathfrak{B})$.

In a short-exact sequence (11), we can identify B_1 with its image under σ_1, and M_1, M_2 under the isomorphism s_1 will be denoted by M. Let $M' = M_3$. Take $\phi: A \to M$ to be a fixed homomorphism, and put $\phi' = s_2\phi: A \to M'$. Then

$$* \to Z^1_\phi(A, \mathfrak{B}_1) \to Z^1_\phi(A, \mathfrak{B}_2) \to Z^1_{\phi'}(A, \mathfrak{B}_3)$$

is an exact sequence of based sets and even of polypi [**1**]. We define $Z^1_{\phi'}(A, \mathfrak{B}_3) \to \boldsymbol{H}^2_\phi(A, \mathfrak{B}_1)$ to be the map which sends each element of $Z^1_{\phi'}(A, \mathfrak{B}_3)$ onto the result of its action (cf. §5) on the ϕ-null class of $\boldsymbol{H}^2_\phi(A, \mathfrak{B}_1)$. With this definition we have

THEOREM 5. *If* (11) *is short exact, then*

$$\begin{aligned} * \to Z^1_\phi(A, \mathfrak{B}_1) \to Z^1_\phi(A, \mathfrak{B}_2) \to Z^1_{\phi'}(A, \mathfrak{B}_3) \to \\ \xrightarrow{\Delta} \boldsymbol{H}^2_\phi(A, \mathfrak{B}_1) \xrightarrow{\Sigma_1} \boldsymbol{H}^2_\phi(A, \mathfrak{B}_2) \xrightarrow{\Sigma_2} \boldsymbol{H}^2_{\phi'}(A, \mathfrak{B}_3) \end{aligned} \tag{15}$$

is exact in the following sense: an element of $Z^1_{\phi'}(A, \mathfrak{B}_3)$ *is in the image of the preceding map if and only if its image under the following map is neutral; an element of* $H^2_\phi(A, \mathfrak{B}_1)$ (resp. $\boldsymbol{H}^2_\phi(A, \mathfrak{B}_2)$) *lies in the image of the preceding map if and only if its image under the following map is ϕ-null* (*resp. neutral*).

It is also possible to characterize the fibres of all maps in (15) and the tricks involved should be incorporated in the notion of *exactness*. The most difficult fibres are those of Σ_2 and, in the case of group extensions, a method has just been described in [6]; it involves a refinement of the spider structure on $H^2(A, \mathcal{B})$.

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