EXTREMAL PROBLEMS IN THE CLASS OF CLOSE-TO-CONVEX FUNCTIONS[1]

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1. **Introduction.** A function $f(z)$, analytic in the open unit disc in the complex z-plane (denoted by D) is said to be *close-to-convex* if there exists a convex univalent function $\phi(z)$ such that $\operatorname{Re}(f'(z)/\phi'(z)) > 0$ for all z in D. In what follows, there is no loss of generality if we assume $f(z)$ to be normalized i.e., $f(0)=0$ and $f'(0)=1$. We can also assume that $\phi(0)=0$ and $|\phi'(0)|=1$. We denote the class of normalized close-to-convex functions by K. It is well known that K is a proper subclass of S—the family of normalized univalent functions in D (see [3]).

In this paper we announce the solutions to two general extremal problems within the class K and, as an application, we announce the rotation theorem for the class K. In the process of solving these extremal problems for the class K, the solutions to these extremal problems for several subclasses of K are found. Some of these solutions are known; we announce the results which do not appear to be known.

2. **Results for close-to-convex functions.** The first problem under consideration is a general coefficient problem. We have the following coefficient theorem for K.

THEOREM 1. *Let $F(z_2, \cdots, z_n)$ be any function having continuous derivatives in each of the $n-1$ variables $z_2, \cdots, z_n$. To each function $f(z) = z + a_2 z^2 + \cdots$ in K associate the number $\operatorname{Re}\{F(a_2, \cdots, a_n)\}$. Then any function $f(z)$ in K which maximizes $\operatorname{Re}\{F(a_2, \cdots, a_n)\}$ over the class K must be of the form*

$$f'(z) = \frac{e^{i\gamma}}{\prod_{j=1}^{M}(1 - ze^{i\alpha_j})^{\mu_j}} \sum_{k=1}^{N} \eta_k \frac{e^{-i\gamma} + ze^{i(\beta_k+\gamma)}}{1 - ze^{i\beta_k}}$$

where

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$$M \leqq n-1,\ N \leqq n-1,\ \mu_j \geqq 0,\ \eta_k \geqq 0,\ \sum_{j=1}^{M} \mu_j = 2,$$

$$\sum_{k=1}^{N} \eta_k = 1,\ -\pi \leqq \alpha_j \leqq \pi,\ -\pi \leqq \beta_k \leqq \pi.$$

We next consider a general extremal problem for K.

THEOREM 2. *Let $F(w)$ be a given entire function and z a given point in D. Then the functional* $\operatorname{Re} F[\log f'(z)]$ *attains its maximum in the class K only for functions of the form*

$$f'(z) = (1 - ze^{i\alpha})^{-2}(1 - ze^{i(\beta+2\gamma)})(1 - ze^{i\beta})^{-1} \tag{1}$$

where $|\gamma| < \pi/2$, $-\pi \leqq \alpha \leqq \pi$ and $-\pi \leqq \beta \leqq \pi$. We exclude from consideration the case in which for the extremal $f(z)$, $F'[\log f'(z)] = 0$.

REMARKS. K is a compact family and hence there are functions in K which maximize each of these functionals. We also note that the final condition in Theorem 2 is satisfied for all "interesting" $F(w)$.

Setting $F(w) = \pm iw$ in Theorem 2 we have the rotation theorem for K.

COROLLARY 1. *The functional* $\arg |f'(z)|$ *attains its maximum in K only for a function of the form* (1). *Thus, for all z in D we have*

$$|\arg f'(z)| \leqq \operatorname*{Max}_{\alpha,\beta,\gamma} |\arg (1 - ze^{i\alpha})^{-2}(1 - ze^{i(\beta+2\gamma)})(1 - ze^{i\beta})^{-1}|$$
$$= 4 \sin^{-1} |z| < 2\pi.$$

3. **Outline of proof.** The class K has the following parametric representation in terms of two Stieltjes integrals:

$$\begin{aligned} f'(z) = e^{i\gamma} \exp\left(-2\int_{-\pi}^{\pi} \log (1 - ze^{it})d\alpha(t)\right) \\ \cdot\left(\cos\gamma \int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\, d\beta(t) - i \sin\gamma\right) \end{aligned} \tag{2}$$

where $\alpha(t)$ and $\beta(t)$ are monotone nondecreasing functions for $-\pi \leqq t \leqq \pi$ and satisfy $\int_{-\pi}^{\pi} d\alpha(t) = \int_{-\pi}^{\pi} d\beta(t) = 1$. Variations within the class K are obtained by varying $\alpha(t)$ and $\beta(t)$ independently within the class of monotone nondecreasing functions on $[-\pi, \pi]$ having total variation 1. This variation is accomplished by means of a method due to G. M. Goluzin [1]. Using these variation formulas for K, we prove that the functions $\alpha(t)$ and $\beta(t)$ which appear in the integral representation for an extremal function must be step functions having at most $n-1$ jumps in the case of Theorem 1 and having one jump in the case of Theorem 2. Evaluating (2) with the given $\alpha(t)$ and $\beta(t)$ we have the stated results. The details will be published elsewhere.

4. Subclasses of K. Let S^* denote the class of normalized starlike functions in D, let C denote the class of normalized convex functions in D and let P' denote the class of normalized functions whose derivative has positive real part in D. S^*, C and P' are subclasses of K (see [3]).

The coefficient theorem for S^* was proved by Hummel [2], and the general extremal problem of Theorem 2 for S^* was solved by Goluzin [1].

The coefficient theorem for C follows from Hummel's result for S^*. We have the following result for C:

THEOREM 3. *Let $F(w)$ be a given entire function and z a given point in D. Then the functional* Re $F[\log f'(z)]$ *attains its maximum in the class C only for functions of the form*

$$f'(z) = (1 - ze^{i\alpha})^{-2}$$

where $-\pi \leqq \alpha \leqq \pi$. We exclude from consideration the case in which for the extremal function $f(z)$, $F'[\log f'(z)] = 0$.

The coefficient theorem for P' follows from the work of M. S. Robertson [4]. We have the following theorem for P'

THEOREM 4. *Let $F(w)$ be a given entire function and z a given point in D. Then the functional* Re$\{F[\log f'(z)]\}$ *attains its maximum in the class P' only for functions of the form*

$$f'(z) = \frac{1 + ze^{i\alpha}}{1 - ze^{i\alpha}} \tag{3}$$

where $-\pi \leqq \alpha \leqq \pi$. We exclude from consideration the case in which for the extremal function $F'[\log f'(z)] = 0$.

Setting $F(w) = \pm iw$ in Theorem 4 we have the rotation theorem for P':

COROLLARY 2. *The functional $|\arg f'(z)|$ attains its maximum in P' only for a function of the form* (3). *Thus,*

$$|\arg f'(z)| \leqq \operatorname*{Max}_{\alpha} \left| \arg \frac{1 + ze^{i\alpha}}{1 - ze^{i\alpha}} \right| = \sin^{-1} \frac{2|z|}{1 + |z|^2} < \frac{\pi}{2}\,.$$

The remarks following Theorem 2 for K apply here for the classes C and P' as well.

5. **Remark.** After seeing the research reported on in this paper, Professor Donald J. Newman pointed out that the bounds on the arguments given above follow from the representation (2). The integrals appearing in (2) are convex combinations of the integrands, and the bounds on the argument of the integrands are easy computations. Of course, the forms of the extremal functions and in particular the sharpness of the rotation theorems do not follow.

References

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