FIBER COBORDISM AND THE INDEX OF A FAMILY OF ELLIPTIC DIFFERENTIAL OPERATORS

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There is a relation between Hirzebruch's differentiable Riemann-Roch theorem [12] of Grothendieck type, and the index formula of Atiyah-Singer [1] for elliptic operators. We shall try to establish such a relation by using a suitable generalization of Thom [14], Conner-Floyd [4] cobordism theory. As a consequence,[1] the analytical index [6] of an elliptic family is computed and shown to be a topological invariant [8] modulo torsion. The author expresses his deep gratitude to H. Cartan, A. Grothendieck, L. Illusie, R. Palais and R. Thom for their indispensable help and kind encouragement.

1. **Notation.** All manifolds M are smooth compact orientable and without boundary. $\Lambda = \boldsymbol{Z}[1/2]$ denotes the subring of the rational numbers $\boldsymbol{Q}$ whose denominators are a power of 2; and

$$k^*_\Lambda(X) = KU^*(X) \otimes z\Lambda, \qquad f^!: \ {}^*_\Lambda(X) \to K^*_\Lambda(Y),$$

denote the natural homomorphism in K-theory induced by a map $f: Y \to X$. If $\pi: \eta \to X$ is an oriented n-dimensional vector bundle over a CW-complex X, we shall denote by $\hat{\pi}: \hat{\eta} \to X$, $q: \hat{\eta} \to \dot{\eta}$, $j: X \to \hat{\eta}$ the associate n-dimensional sphere bundle over X, its projection onto the Thom space $\dot{\eta}$ and the embedding induced by the zero section of η.

For a finite CW-complex B, C_B is the category of all fiber bundles $p: X \to B$, over B, with fiber an orientable manifold M and structural group a subgroup of all orientation preserving diffeomorphisms of M. We shall denote by

$$\pi: T_\natural(X) \to X$$

the cotangent bundle along the fiber of $p: X \to B$. A map $f: X \to Y$ in C_B is supposed to be fiber preserving and C^∞ along each fiber. If ξ is a complex vector bundle over X, $J^k_\natural(X) \to X$, $k \in \boldsymbol{Z}_+$, will denote the k-jet bundle [11] along the fiber, characterized by the restriction of $J^k_\natural(X)$ on a fiber M is the k-jet bundle of the restriction of ξ on M. In a way similar to that of Palais [9] the *Sobolev chain of* ξ is well defined: $H^s_\natural(\xi) \to B$, $s \in \boldsymbol{R}$, as Hilbertian vector bundles over B whose fibers are the Sobolev chain $H^s(\xi | M)$. Now if η is another complex

[1] This gives an answer to a problem of Chern (Bull. Amer. Math. Soc. **72** (1966), 167–219), in fact we can obtain thus a Riemann-Roch theorem of Grothendieck for the Chern character of the direct image of the sheaf of holomorphic sections of a locally trivial holomorphic vector bundle.

vector bundle over X, then a bundle map: $D \in \mathrm{Hom}(J^k_\natural(\xi), \eta)$ is called a kth *order family of differential operator* [11] *on X parametrized by B.* And it is clear that D induces a vector bundle map, $D^s: H^s_\natural(\xi) \to H^{s-k}_\natural(\eta)$. Let $T^{\cdot}_\natural(X)^0$ denote the nonzero vectors of $T_\natural(X)$; $\xi^\#$, $\eta^\#$ the pull back bundles over $T_\natural(X)^0$ and $D^\#$ the induced symbol map from $\xi^\#$ into $\eta^\#$. When $D^\#$ is an isomorphism, D is called *elliptic* and its symbol class $\sigma(D) \in K(T^{\bullet}_\natural(X))$, is defined by the ordinary difference construction of K-theory. Now the ellipticity of D implies that D^s is a Fredholm[2] map of a Hilbert bundle. Hence using Illusie's difference construction for Hilbertian bundles [6], a generalization of Jänich's theorem [7], D^s determines an element

$$i_a(D) \in K(B)$$

the *analytic index of the elliptic family D.*

2. **Differentiable Riemann-Roch theorem for manifold with boundary.** Let $\overline{C}_B$ be the category of fiber bundle, $p: W \to B \in \overline{C}_B$, whose fiber is a compact smooth manifold with boundary. It is clear that the boundary points ∂W form again a fiber bundle $p: \partial W \to B \in C_B$; over B; a functor $\partial: \overline{C}_B \to C_B$ is well defined. By a map in $\overline{C}_B$, $f: W \to W'$ we shall mean a bundle map such that $f(\partial W) \subseteq \partial W'$. Then using the same notation as in [12] we have

THEOREM. *Given a characteristic class $\alpha \in \mathrm{Hom}(KSO, L)$, where $L = K_\Lambda$ or H^{2*}, and a map $f: W \to W'$ in $\overline{C}_B$, there exist unique homomorphisms $f^\alpha_!$ and $p^\alpha_!$ such that the diagrams*

$$\begin{tikzcd} L^*(W, \partial W) \arrow[rr, "f^\alpha_!"] \arrow[dr, "p^\alpha_!"'] & & L^*(W', \partial W') \arrow[dl, "p^\alpha_!"] \\ & L^*(B) & \end{tikzcd} \qquad \begin{tikzcd} L^*(\partial W) \arrow[rr, "\delta"] \arrow[dr, "p^\alpha_!"'] & & L^*(W, \partial W) \arrow[dl, "p^\alpha_!"] \\ & L^*(B) & \end{tikzcd}$$

are commutative, where δ is the coboundary morphism and the left hand $p^\alpha_!$ is defined in [12].

3. **Fiber cobordism.** Let B be a fixed connected finite CW-complex and (Y, Y') a pair of topological spaces. An *oriented B-singular n-manifold of (Y, Y')* is a pair (X, f) where

$p: X \to B \in \overline{C}_B$ with fiber n-dimensional manifold,

$f: X \to Y$ a continuous map which satisfies $f(\partial X) \subseteq Y'$.

[2] That is the "operator with index" in 3 which is different from the classical definition of Fredholm operators given by Grothendieck.

If $Y'=\emptyset$, then $p: X\to B$ is of course in C_B. An oriented B-singular manifold (X, f) is said to be a boundary if and only if there exists $g: W\to B\in\overline{C}_B$ and $g: W\to Y$ such that: $X\subseteq\partial W$, $g|X=f$ and $g(\partial W-X)\subseteq Y'$. This gives [4] using disjoint sum, an equivalence relation between B-singular manifolds, and hence an abelian group is thus obtained

$$\Omega_*^B(Y, Y') = \Sigma\Omega_n^B(Y, Y'),$$

which is a *bifunctor* covariant in (Y, Y') and contravariant in B.

THEOREM. *When B is fixed, $\Omega_*^B(Y, Y')$ is an homology theory.*

It is pointed out by D. B. A. Epstein and R. Thom that Ω_*^B *is not a cohomology theory* with respect to B. This comes from the close relation between Ω_*^B and the homotopy type of the group of diffeomorphisms. Let us denote by $C_t^\infty(S^{n+k}, MSO(k))\subseteq C^0(S^{n+k}, MSO(k))$ the subspace of the space of all continuous maps from the $n+k$ dimensional sphere into the universal Thom space which are smooth wherever differentiability makes sense and transverse regular on the k-dimensional Grassman manifold $G_{k,m}$ for m large. Similarly, let $C_{t,1}^\infty(S^{n+k}\times I, MSO(k))$, $I=[0, 1]\subseteq R$ be those homotopies h, transverse on $G_{k,m}$ and such that projection of $h^{-1}(G_{k,m})\subseteq S^{n+k}\times I$ into I has only isolated nondegenerated critical points. And we define the *strong homotopy* of maps in $C^0(B, C^\infty(S^{n+k}MSO(k)))$ by the existence of a homotopy in $C^0(B, C_{t,1}^\infty(S^{n+k}\times I, MSO(k)))$. Then we have

THEOREM. *For $k>n+\dim B+2$ we have an isomorphism*

$$\Omega_n^B = \Omega_n^B(\text{point}) \approx \pi_{0,t}(C^0(B, C_t^\infty(S^{n+k}, MSO(k)))$$

where $\pi_{0,t}$ denote the strong classes of maps.

For the general case one can use the $\times$ product, hence there is a canonical homomorphism $\Omega_*^B(Y)\to\pi_*^B(Y\times MSO)$ to the Barratt group. Pontrjagin and Stiefel-Whitney numbers are well defined in $\Omega_*^B(Y, Y')$ associated to elements of $H^*(B)\otimes H^*(Y, Y')$. Other properties in [4] can be also carried out. In particular we have the "Whitney sum": $\Omega_*^B\otimes\Omega_*^B(Y, Y')\to\Omega_*^B(Y, Y')$; and when Y is a H-space $\Omega_*^B(Y)$ is a graded algebra. Several generalizations can be done in a similar way: G-space, replace bundle by singularity type, or Y by a bundle over B etc. We remark, in fact there *exist many geometrically well-defined functors which are neither cohomology theories nor representable functors* in the ordinary sense.

4. **The homomorphism α_1.** We shall denote by U the inductive limit

$\lim U(n)$ of unitary groups and $F = BU \times Z$; where BU is the classifying space of U: the space of Fredholm operators [7]. Let B denote a finite CW-complex and $\alpha \in \operatorname{Hom}(KSO, K_\Lambda)$ [9] be a characteristic class. We shall define a homomorphism

$$\alpha_!: \overset{B}{\Omega_*}(F) + \overset{B}{\Omega_*}(U) \to \overset{*}{K_\Lambda}(B)$$

as follows. If $[X, f]$ is a B-singular manifold of F or U, represented by $p: X \to B$, $f: X \to F$, (or U). Then $\alpha_!([X, f]) = p_!^\alpha([f])$, where $p_!^\alpha$ is the homomorphism [12] in §2 and $[f] \in K^*(X)$ the class determined by f. Using the theorem in §2 we deduce easily that $\alpha_!$ is well defined, and *is a homomorphism of Z_2-graded algebra over the Z_2 graded ring* $K^*(B)$.

We introduce on $K_\Lambda^*(B)$ a *Ω_*^B-module structure* as follows. Recall that a B-singular manifold $[X, p] \in \Omega_*^B$ (point) $= \Omega_*^B$ is represented by a bundle over B; $p: X \to B$. Then we define $\bar{\alpha}_!: \Omega_*^B \otimes K_\Lambda^*(B) \to K_\Lambda^*(B)$ by $\bar{\alpha}_!([X, p] \otimes \beta) = p_!^\alpha(1) \cdot \beta$, $\beta \in K_\Lambda^*(B)$. Identify $\Omega_*(F) = \Omega_*^{\text{point}}(F)$ as a subgroup of $\Omega_*^B(F)$ using the projection of B into the point, we have

THEOREM. *Consider $\Omega_*^B(F) + \Omega_*^B(U)$ and $K_\Lambda^*(B)$ both as modules over the ring Ω_*^B, then the map: $\alpha_!: \Omega_*^B(F) + \Omega_*^B(U) \to K_\Lambda^*(B)$ is a Ω_*^B-homomorphism. Moreover any Ω_*^B-homomorphism*

$$\overset{B}{\Omega_*}(F) + \overset{B}{\Omega_*}(U) \to \overset{*}{K}(B) \otimes \boldsymbol{Q}$$

whose restriction on $\Omega_*(F) + \Omega_*(U)$ equals that of $\alpha_!$ is identically equal to $\alpha_!$.

5. **The index formula.** Let $p: X \to B \in C_B$ and D a family of elliptic differential operators on X. Using the notations in §1, we define [11] the *character* of D

$$ch(D) = \hat{\pi}_! q^!(\sigma(D)) \in \overset{*}{K_\Lambda}(X)$$

to be the image of the symbol class of D, $\sigma(D) \in K^*(T_\natural^\bullet(X))$ by the composite map $K_\Lambda^*(T_\natural^\bullet(X)) \to^{q^!} K_\Lambda^*(T_{\hat{\natural}}(X)) \to \pi_! K_\Lambda^*(X)$ where $\pi_!$ corresponds to the trivial characteristic class and π the projection of the cotangent bundle along the fiber $T_\natural(X)$ on X. Now let $\alpha_0 \in \operatorname{Hom}(KSO, K_\Lambda)$ be the characteristic class determined as follows: for each two-dimensional oriented real vector bundle η we have $(\alpha_0(\eta))^- = 2 + \xi + \bar{\xi}$ where ξ is the unique complex line bundle isomorphic to η and $\bar{\xi}$ is its conjugate. Then we define the topological index of D [11] by

$$i_t(D) = p_!^{\alpha_0}(ch(D)) \in K_\Lambda^*(B).$$

When B is a point, the topological index of an elliptic operator is by definition in $\mathbf{Z}[1/2]$. We have the following index theorem.

THEOREM. *Let $p: X \to B$ be a fiber bundle over a finite* CW-*complex B, and D is a family of elliptic differential operators on X parametrized by B, then the two indexes of D are equal modulo torsions*: i.e., $i_\alpha(D) = p_!^{\alpha_0}(ch(D)) = i_t(D)$ *in* $K^*(B) \otimes \mathbf{Q}$.

Our proof follows closely to that of Atiyah-Singer and Palais [**9**]. We first show that each index defines a homomorphism from $\Omega_*^B(F)$ into $K_\Lambda(B)$. If $p: X \to B \in C_B$, ξ η complex vector bundles on X, one defines $Op^k(\xi, \eta)$ to be those continuous bundle maps T: $C^\infty(\xi) \to C^\infty(\eta)$ which can be extended to a map from $H_\natural^\gamma(\xi)$ into $H_\natural^{\gamma-k}(\eta)$. And $\mathrm{Int}^k(\xi, \eta) \subseteq Op^k(\xi, \eta)$ are those T such that on each fiber of the bundle $C^\infty(\xi) \to B$, T is in Int^k $(\xi_b; \eta_b)$ where $b \in B$, $\epsilon_b = \xi | p^{-1}(b)$. Then the properties of Seeley algebra in [**9**] can be extended to this case because of the neutrality of Palais' construction. That the analytic index of a boundary is zero has been communicated to the author by L. Illusie using a method similar to [**3**]. So the analytic homomorphism from $\Omega_*^B(F)$ into $K(B)$ is thus obtained. The topological one follows from its definition and $\alpha_{0!}$. The two homomorphisms satisfy a condition analogous to those in the case of a single operator (cf. §4) and [**9**]. For the topological one this is just the Grothendieck formulas [**12**] for Riemann-Roch Theorem. Then we use Illusie's [**6**] results on the complexes of Hilbertian bundle; its equivalence with the ordinary ones for the difference construction as well as product, and follow the indication in [**3**], [**9**] to obtain the condition for the analytic case. Then using the results in §4 we obtain the uniqueness of such homomorphisms.

6. **Special case.** Let θ be a characteristic class in K-theory, and D a family of elliptic operators defined on $p: X \to B \in C_B$. Then D is said to be *θ-universal*, if its symbol class $\sigma(D) \in K(T_\natural^*(X))$ satisfies the relation: $j^!(\sigma(D)) = \theta(T_\natural(X))$ in $K(X)$ where $j^!$ is the homomorphism induced by the zero section of the cotangent bundle along the fiber $T_\natural(X)$.

EXAMPLE. Suppose the fiber of $p: X \to B$ is an even dimensional manifold (resp. a complex analytic manifold and the structural group of p consists of biholomorphic maps). Then we have the following families of elliptic differential operators on X parametrized by

B^8: $(D_0)_\natural$, $(d+\delta)_\natural$, (resp. $(\bar\partial+\bar\partial^*)_\natural$) whose restriction to each fiber of $C^\infty(\xi^+)\to B$ are respectively the Hirzebruch operator and $d+\delta$ associated to a Riemann structure (resp. $\bar\partial+\bar\partial^*$), where ξ^+ denotes the suitable vector bundle for these operators. These operators are respectively λ^0_-, $\lambda_{-1}\otimes \mathbf{C}$, (resp. λ_{-1}) *universal.* We denote by λ^0_- the characteristic class determined by $\lambda^0_-(\eta)=\xi-\bar\xi$ for each 2-dimensional real oriented bundle η, where ξ is the unique complex line bundle isomorphic to η, and $\bar\xi$ its conjugate. λ_{-1} is the Grothendieck alternating sum [2] of exterior algebra of complex vector bundles, and $\lambda_{-1}\otimes\mathbf{C}(\eta)=\lambda_{-1}(\eta\otimes\mathbf{C})$ for each real bundle η.

We recall [12] that if $\mathrm{Hom}(E_C, K_\Lambda)_0$ (resp. $\mathrm{Hom}(E^0_R, K_\Lambda)_0$) denotes the sub semigroup of those characteristic classes θ for complex vector bundles (resp. even dimensional oriented real bundles) which vanish on the 1-dimensional trivial bundle $\theta(1_C)=0$ (resp. $\theta(2_R)=0$). Then there is a *canonical bijection* τ: $\mathrm{Hom}(E^0_C, K_\Lambda)_0\to\mathrm{Hom}(E^0_C, K_\Lambda)$ which commutes with the canonical inclusion: $\mathrm{Hom}(E^0_R,K_\Lambda)_0\to\mathrm{Hom}(E_C,K_\Lambda)_0$. We shall denote by $\tau(\theta)$ the image of θ (verifying $\theta(1_C)=\theta(2_R)=0$) by τ.

THEOREM. *Suppose $p: X\to B\in C_B$ with even dimensional fiber and D a family of elliptic operators on X, which is θ-universal $\theta\in\mathrm{Hom}(E,$ $K_\Lambda)_0$, $E=E_C$ or E^0_R. Then the topological index of D is given by*

$$i_t(D) = p_!^{\alpha_0}(\tau(\theta)(T_\natural(X))),$$

hence it depends only the value of $\tau(\theta)$ on the cotangent bundle along the fiber.

Using Hirzebruch's Riemann-Roch theorem [12], we obtain the Chern character of the index of those operators in the above example as follows:

$$\mathrm{ch}(i(D_0)_\natural) = p_*(L(T_\natural(X)), \qquad \mathrm{ch}(i(d+\delta)_\natural) = p_*(\chi(T_\natural(X)),$$
$$\mathrm{ch}(i(\bar\partial+\bar\partial^*)_\natural) = p_*(\mathrm{Todd}(T_\natural(X)))$$

where $p_*: H^*(X, Q)\to H^*(B, Q)$ is the Gysin homomorphism [12], and L the L-genus corresponding to the formal power series: $-t/\tanh(t/2)$. Similarly we obtain the case with coefficient bundle for universal operators.

7. **The index function.** Let p: $X\to B\in C_B$ with fiber even (resp. odd) dimensional manifold. We recall [9] that the index function

[8] Compare with K. Kodaira-D. C. Spencer, *On deformation of complex analytic structures.* III, Ann. of Math, 1948.

$i\colon K(X) \to K_\Lambda(B)$ (resp. $K(X) \to K_\Lambda(B)$) is defined by $i(x) = i_t((D_0)_\natural \otimes 1_\eta)$ where η is a complex vector bundle whose stable class is x, and $(D_0)_\natural$ the Hirzebruch operator of p, and we denote by i_Q the rational index function. Similarly we obtain the odd dimensional case, and we have

THEOREM. *The rational index function i_Q is equal to the composite map*

$$p_!^{\alpha_0} \otimes \boldsymbol{Q}\colon K^*(X) \to K_\Lambda^*(X) \xrightarrow{p_!^{\alpha_0}} K_\Lambda^*(B) \to K^*(B) \otimes \boldsymbol{Q},$$

and it is independent of the underlying differentiable structure of X.

This follows from Novikov's [8] invariance of rational Pontrjagin classes, because the corresponding Wu class: $\mathrm{Wu}(\alpha_0, 1)$ [12] is a function of Pontrjagin classes. However, the *index function "i"* itself is *not* known to be dependent on the differentiable structure of X. In fact, even if the integer Pontrjagin classes are topological invariants, our method cannot affirm that "i" is a topological invariant. One might expect that for each characteristic class $\alpha \in \mathrm{Hom}(KSO, K_\Lambda)$ the corresponding Gysin homomorphism [12] modulo torsion

$$p_!^\alpha\colon K^*(X) \to K^*(B) \otimes \boldsymbol{Q}$$

is a function of Pontrjagin classes, hence a topological invariant. But this is *not true*. In fact, there are many α for which $p_!^\alpha$ cannot be determined by Pontrjagin classes. Hence the interpretations of those $p_!^\alpha$ by analytic invariants like the index of elliptic operators, may be of some interest to study.

8. **Remarks.** The Gysin homomorphism for a manifold with boundary indicates that a generalization of the index theorem for a manifold with boundary to $\overline{C}_B$ perhaps exists. Using $\Lambda = \boldsymbol{Z}[1/2]$ to kill all two torsion seems to imply the existence of some modulo 2 invariant of elliptic operator which corresponds to the torsion group of cobordism. One can also study the case for G-space given by G. B. Segal (G a compact Lie group using the characteristic classes in K_G-theory developed by Nguyen Dinh Ngoc (to be published)).

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