A PROPERTY OF THE L_2-NORM OF A CONVOLUTION

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Communicated by A. Zygmund, December 8, 1965

Introduction. It is known that the convolution of two members, f and g, of $L_2(-\infty, +\infty)$ can be a null function without either f or g being a null function. But, if one defines f_ν by setting $f_\nu(x) = e^{i\nu x}f(x)$ for all x, f_ν and g will have a convolution that is not a null function for a suitable choice of ν. There is apparently no information available on how the L_2-norm of the latter convolution depends on ν.

A partial answer to this problem will be provided in the present paper. There will be derived a lower bound on the supremum in ν of the L_2-norm of the convolution of f_ν and g. The lower bound will be expressed in terms of a notion of ϵ-approximate support which is an $L_1(-\infty, +\infty)$ analog of the concept of support of a continuous function on a locally compact space. The inequality will be shown to be sharp in the sense that one can construct an f and a g for which the lower bound is approached arbitrarily closely.

Definitions and notation. Because of the need for uniqueness and because of the nature of the L_1-norm, an appropriate analog for $L_1(-\infty, +\infty)$ of the notion of support is the following.

DEFINITION. *The ϵ-approximate support of a member f of $L_1(-\infty, +\infty)$ is defined to be the closed interval $I_{\epsilon,f}$ such that*

(a) *$I_{\epsilon,f}$ is symmetric about the smallest real number x_0 for which*

$$\int_{-\infty}^{x_0} |f(x)|\, dx = (\tfrac{1}{2})\|f\|_1,$$

(b) $\int_{I_{\epsilon,f}} |f(x)|\, dx = (1-\epsilon)\|f\|_1$,

$\|f\|_1$ *being the L_1-norm of f.* The existence and uniqueness of x_0 and $I_{\epsilon,f}$ are clear from the absolute continuity of the indefinite integral of $|f|$.

For any Lebesgue-measurable set E the measure of E is denoted by $m(E)$ and the characteristic function is denoted by $\chi(E)$. Given any two measurable functions on the real numbers, f and g, such that for almost all x, $f(y)g(x-y)$ is in $L_1(-\infty, +\infty)$ one denotes by $f * g$ the function for which $(f * g)(x) = \int_{-\infty}^{+\infty} f(y)g(x-y)dy$ a.e. Given any f in $L_1(-\infty, +\infty) \cap L_2(-\infty, +\infty)$ one defines the Fourier transform of f, denoted by $\hat{f}$, by requiring that for all real ω, $\hat{f}(\omega) = (2\pi)^{-1/2}\int_{-\infty}^{+\infty} \exp(-i\omega x)f(x)dx$. Thus, the definition of $\hat{f}$ for an arbitrary f in $L_2(-\infty, +\infty)$ is determined.

Results. One lemma is required for proof of the principal result. It appears below.

LEMMA. *Given any two nonnegative, nonnull functions h and k in* $L_1(-\infty, +\infty)$ *such that* $\|h\|_1 = \|k\|_1 = 1$, *then*

$$\sup_{-\infty < x < +\infty} (h * k)(x) \geqq \sup_{0 < \epsilon < 1} (1-\epsilon)^2 [m(I_{\epsilon,h}) + m(I_{\epsilon,k})]^{-1}. \tag{1}$$

PROOF. For any real ϵ such that $0 < \epsilon < 1$, we define h_ϵ and k_ϵ, nonnegative and nonnull members of $L_1(-\infty, +\infty)$, by the equations below.

$$h_\epsilon(z) = \chi_{I_{\epsilon,h}}(z) h(z), \qquad \text{all } z, \tag{2}$$

$$k_\epsilon(x) = \chi_{I_{\epsilon,k}}(z) k(z), \qquad \text{all } z. \tag{3}$$

First, one can use (2) and (3) to write:

$$\begin{aligned} m(I_{\epsilon,h}) + m(I_{\epsilon,k}) &= m\{[x \mid I_{\epsilon,h} \cap (x - I_{\epsilon,k}) \neq \emptyset]\} \\ &= m\{[x \mid \exists y \ni y \in I_{\epsilon,h},\ x - y \in I_{\epsilon,k}]\} \\ &\geqq m\{[x \mid (h_\epsilon * k_\epsilon)(x) \neq 0]\}. \end{aligned} \tag{4}$$

Then since $k_\epsilon(x)h_\epsilon(y)$ belongs to $L_1[(-\infty, +\infty)X(-\infty, +\infty)]$, one can combine (4) with the Fubini theorem for multiple integrals and well-known properties of the transformation T defined by $T(x, y) = (x-y, y)$ to write the following sequence of equalities.

$$\begin{aligned} &[m(I_{\epsilon,h}) + m(I_{\epsilon,k})]\left[\sup_{-\infty<x<+\infty} (h*k)(x)\right] \\ &\qquad \geqq m(\{x \mid (h_\epsilon * k_\epsilon)(x) \neq 0\})\left[\sup_{-\infty<x<+\infty} (h*k)(x)\right] \\ &\qquad \geqq m\{[x \mid (h_\epsilon * k_\epsilon)(x) \neq 0]\}\left[\sup_{-\infty<x<+\infty} (h_\epsilon * k_\epsilon)(x)\right] \\ &\qquad \geqq \int_{-\infty}^{+\infty} (h_\epsilon * k_\epsilon)(x)\, dx \\ &\qquad = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} [h_\epsilon(y) k_\epsilon(x-y)]\, dxdy \\ &\qquad = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} [h_\epsilon(y) k_\epsilon(x)]\, dxdy \\ &\qquad = \left(\int_{-\infty}^{+\infty} h_\epsilon(y)\, dy\right)\left(\int_{-\infty}^{+\infty} k_\epsilon(x)\, dx\right) \\ &\qquad = (1-\epsilon)^2. \end{aligned} \tag{5}$$

The conclusion of this lemma follows directly from (5).

This lemma permits one to prove the following theorem.

THEOREM. *Let f and g be any two members of $L_2(-\infty, +\infty)$ such that $\|f\|_2 = \|g\|_2 = 1$. Let $f_\nu(x) = e^{i\nu x}f(x)$ for all x. Let $F(x) = |\hat{f}(x)|^2$ and $G(x) = |\hat{g}(x)|^2$ for all x. Then*

$$\sup_{-\infty<\nu<+\infty} \|f_\nu * g\|_2 \geqq \sup_{0<\epsilon<1} (1-\epsilon)\{2\pi[m(I_{\epsilon,F}) + m(I_{\epsilon,G})]^{-1}\}^{1/2}. \tag{6}$$

The inequality (6) is sharp in the sense that for every positive number η there are choices of f and g for which the right side of the inequality is finite and for which the ratio of the expression on the left-hand side of (6) to the expression on the right-hand side exceeds 1 by less than η.

PROOF. The lemma and the Plancherel theorem combined yield (6).

To prove the rest of the theorem, let η be a fixed but arbitrary positive number. Then two members, p and q, of $L_2(-\infty, +\infty)$ will be defined and shown to have the asserted properties relative to η. These functions will be defined in terms of their Fourier transforms.

$$\hat{p}(\omega) = \begin{cases} (\pi\Delta)^{-1/2}(1 + i\omega)^{-1}, & |\omega| \leqq \tan \dfrac{\pi}{2}\Delta, \\ 0, & |\omega| > \tan \dfrac{\pi}{2}\Delta, \end{cases} \tag{7}$$

$$\hat{q}(\omega) = \begin{cases} \pi^{-1/2}(1 + i\Delta\omega)^{-1}, & |\omega| \leqq \Delta^{-1}\tan \dfrac{\pi}{2}\Delta, \\ 0, & |\omega| > \Delta^{-1}\tan \dfrac{\pi}{2}\Delta. \end{cases} \tag{8}$$

Here Δ is assumed to be positive and less than 1. Then, with the aid of the definitions of $\hat{p}$ and $\hat{q}$ and the Plancherel theorem, it can be seen that one has:

$$\begin{aligned} \sup_{-\infty<\nu<+\infty} \|p_\nu * q\|_2 &= \sup_{-\infty<\nu<+\infty} \left[2\pi \int_{-\infty}^{+\infty} |\hat{p}(\omega-\nu)|^2 |\hat{q}(\omega)|^2 d\omega\right]^{1/2} \\ &= \left[2\pi \int_{-\infty}^{+\infty} |\hat{p}(\omega)|^2 |\hat{q}(\omega)|^2 d\omega\right]^{1/2}. \end{aligned} \tag{9}$$

And the latter integral has the following evaluation.

$$\int_{-\infty}^{+\infty} |\hat{p}(\omega)|^2 |\hat{q}(\omega)|^2 d\omega$$

$$= \left(2\Delta^{-1}\tan\frac{\pi}{2}\Delta + 2\tan\frac{\pi}{2}\Delta\right)^{-1}\left(\frac{\tan \pi\Delta/2}{\pi\Delta/2}\right) \tag{10}$$

$$\cdot\left(\frac{1 - 2/\pi \tan^{-1}(\Delta \tan \pi\Delta/2)}{1-\Delta}\right).$$

However, one can see:

$$m(I_{0,P}) = 2\tan\frac{\pi}{2}\Delta, \tag{11}$$

$$m(I_{0,Q}) = 2\Delta^{-1}\tan\frac{\pi}{2}\Delta \tag{12}$$

where P and Q are determined by setting $P(\omega) = |\hat{p}(\omega)|^2$ and $Q(\omega) = |\hat{q}(\omega)|^2$ for all ω. Thus, there results:

$$\left(2\Delta^{-1}\tan\frac{\pi}{2}\Delta + \tan\frac{\pi}{2}\Delta\right)^{1/2} \tag{13}$$

$$\leqq \sup_{0<\epsilon<1}(1-\epsilon)\{2\pi[m(I_{\epsilon,P}) + m(I_{\epsilon,Q})]^{-1}\}^{1/2}.$$

Hence, combining (6), (9), (10), and (13), one can conclude that when Δ is small enough for

$$\left[\frac{\tan\frac{\pi}{2}\Delta}{\pi\Delta/2}\cdot\frac{1-\frac{2}{\pi}\tan^{-1}\left(\Delta\tan\frac{\pi}{2}\Delta\right)}{1-\Delta}\right]^{1/2} - 1$$

to be less than η, then the same is true of

$$\left(\sup_{-\infty<\nu<+\infty}\|p_\nu * q\|_2\right) \Big/ \left\{\sup_{0<\epsilon<1}\frac{(1-\epsilon)(2\pi)^{1/2}}{[m(I_{\epsilon,P}) + m(I_{\epsilon,Q})]^{1/2}}\right\} - 1.$$

It is, of course, clear from (9) and (10) that $\sup_{-\infty<\nu<+\infty}\|p_\nu * q\|_2$ is finite.

Thus, the second part of the theorem has been approved.

COROLLARY. *Let the notation of the theorem hold. Further, let f and g be restrictions to $(-\infty, +\infty)$ of entire functions of exponential type such that the types of f and g are E_1 and E_2, respectively. Then*

$$\sup_{-\infty<\nu<+\infty}\|f_\nu * g\|_2 \geqq \left[\frac{\pi}{2}(E_1+E_2)^{-1}\right]^{1/2}.$$

PROOF. As indicated by Theorem 21 [1] the transforms of f and g vanish outside $[-E_1, E_1]$ and $[-E_2, E_2]$ respectively. Thus,

$$m(I_{o,F}) \leqq 4E_1, \tag{14}$$

$$m(I_{o,G}) \leqq 4E_2. \tag{15}$$

Since the indefinite integrals of $|f|$ and $|g|$ are absolutely continuous, (14) and (15) permit the following inequality.

$$\sup_{0<\epsilon<1} (1-\epsilon)\{2\pi[m(I_{\epsilon,F}) + m(I_{\epsilon,G})]^{-1}\}^{1/2} \geqq \{2\pi[4E_1 + 4E_2]^{-1}\}^{1/2}. \tag{16}$$

The assertion of the corollary follows from (16) and the theorem.

REFERENCE

1. R. E. A. C. Paley and N. Wiener, *Fourier transforms in the complex domain*, Amer. Math. Soc. Colloq. Publ., Vol. 19, Amer. Math. Soc., Providence, R. I., 1934.

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