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A RELATION BETWEEN A THEOREM OF BOHR AND SIDON SETS

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Communicated by W. Rudin, December 17, 1965

1. **Introduction.** In 1913, Bohr [1] proved the following theorem for Dirichlet series: if

$$f(\sigma + it) = \sum_{n=1}^{\infty} c(n)n^{-\sigma-it} \tag{1}$$

and if $|f(\sigma+it)| \leqq 1$ for all $\sigma > 0$, then

$$\sum_{p} | c(p) | \leqq 1, \tag{2}$$

the sum in (2) extending over all primes.

A set of positive integers E will be called a *Bohr set* if there is a finite constant B such that for every function f as in (1)

$$\sum_{n\in E} | c(n) | \leqq B. \tag{3}$$

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[1] This research was supported in part by Air Force Office of Scientific Research Grant AF-AROSR 335-63.

It is easily seen that E is a Bohr set if and only if for every finite sum $f(t) = \sum c(n)n^{-it}$

$$\sum_{n\in E} |c(n)| \leqq B \sup_{-\infty<t<\infty} |f(t)|. \tag{4}$$

Let G be a compact Abelian group and E a subset of its dual group Γ. An *E-polynomial* is a trigonometric polynomial, F, such that $\hat{F}(\gamma) = 0$ for $\gamma \notin E$ where

$$\hat{F}(\gamma) = \int_G F(x)\gamma(-x)\, dx, \qquad \gamma \in \Gamma.$$

Here dx is normalized Haar measure on G. E is called a *Sidon set* if there is a finite constant B such that

$$\sum_{\gamma\in E} |\hat{F}(\gamma)| \leqq B \sup_{x\in G} |F(x)| = B\|F\|^\infty \tag{5}$$

for every E-polynomial F.

Let T^ω be the direct product of a countably infinite collection of circles. T^ω is a compact Abelian group with dual group Z^ω. Each $\gamma \in Z^\omega$ is given by a sequence of integers $\{\alpha_k\}$ where only a finite number of the α_k are not zero. $M(T^\omega)$ is the space of regular Borel measures, μ, on T^ω with finite total variation $\|\mu\|$. $\hat{\mu}$ is the Fourier-Stieltjes transform of μ.

In this note we give a characterization of Bohr sets in terms of Sidon sets in Z^ω and certain measures on T^ω. It is then possible to obtain a sufficient arithmetic condition for Bohr sets.

2. **The relation between Bohr sets and Sidon sets.** P will denote the positive cone of Z^ω. Let $p_1, p_2, \cdots$ be the primes. If n is an integer and $n = \prod p_j^{\alpha_j}$, then we associate n with the element $\gamma_n = (\alpha_1, \alpha_2, \cdots)$ of P. For a set of positive integers E, $\hat{E} = \{\gamma_n : n \in E\}$.

To a function $f(t) = \sum c(n)n^{-it}$ we associate the function $F(x) = \sum c(n)\gamma_n(x)$ on T^ω. Bohr noticed the following: if $\phi: (-\infty, \infty) \to T^\omega$ by

$$\phi(t) = (\exp(-it \log p_1), \exp(-it \log p_2), \cdots) \tag{6}$$

then $\gamma_n(\phi(t)) = n^{-it}$ so that $F(\phi(t)) = f(t)$. Now since $\{\log p_j\}$ is linearly independent over the integers, $\phi(-\infty, \infty)$ is dense in T^ω. Thus

$$\|F\|_\infty = \sup_{-\infty<t<\infty} |f(t)|. \tag{7}$$

THEOREM. *A set of positive integers E is a Bohr set if and only if*

(a) *$\hat{E}$ is a Sidon set in T^ω, and*

(b) *there is a measure $\mu \in M(T^\omega)$ such that*

$$\begin{aligned} \hat{\mu}(\gamma) &= 1 \quad \text{if } \gamma \in \hat{E}, \\ &= 0 \quad \text{if } \gamma \in P - \hat{E}. \end{aligned} \tag{8}$$

PROOF. Let $F(x) = \sum \hat{F}(\gamma_n)\gamma_n(x)$ be a P-polynomial and let $f(t) = \sum \hat{F}(\gamma_n)n^{-it}$.

If E is a Bohr set then by (7)

$$\sum_{\gamma \in \hat{E}} |\hat{F}(\gamma)| = \sum_{n \in E} |\hat{F}(\gamma_n)| \leqq B \sup_{-\infty < t < \infty} |f(t)| = B\|F\|_\infty. \tag{9}$$

Thus if b is a function on $\hat{E}$ and $|b(\gamma)| \leqq 1$ then $L(F) = \sum_{\gamma \in \hat{E}} b(\gamma)\hat{F}(\gamma)$ is a bounded linear functional on the P-polynomials with norm at most B. By the Hahn-Banach and Riesz representation theorems there is a measure $\mu \in M(T^\omega)$ with

$$\begin{aligned} \hat{\mu}(\gamma) &= b(\gamma), \qquad \gamma \in \hat{E}, \\ &= 0, \qquad\quad \gamma \in P - \hat{E}. \end{aligned}$$

By [3, Theorem 5.7.3], $\hat{E}$ is a Sidon set; by taking $b \equiv 1$ we obtain the measure for (b).

Conversely suppose (a) and (b) are true for E and let $f(t) = \sum c(n)n^{-it}$ be a finite sum. By (a) and the proof of [3, Theorem 5.7.3] there is $\nu \in M(T^\omega)$ with $\|\nu\| \leqq B$ (B depends only on E) and $c(n)\hat{\nu}(\gamma_n) = |c(n)|$ for $n \in E$. Let μ be as in (b) and $*$ denote ordinary convolution. Then

$$\begin{aligned} \sum_{n \in E} |c(n)| &= \mu * \nu * \sum c(n)\gamma_n(0) \\ &\leqq B\|\mu\|\|F\|_\infty \\ &\leqq B' \sup_{-\infty < t < \infty} |f(t)|. \end{aligned}$$

COROLLARY. *Let $E = \{n_1, n_2, \cdots\}$ be a set of positive integers satisfying*

(c) *$\{\log n_j\}$ are linearly independent over the integers, and*

(d) *if n is a positive integer, $\{\beta_j\}$ is a collection of integers, $\sum \beta_j = 1$, and $n = \prod n^{\beta_j}$ then $n \in E$.*

Then E is a Bohr set.

PROOF. It follows from (c) that if $k_1 < k_2 < \cdots < k_s$ then $0 \neq \pm\gamma_{n_{k_1}} \pm\gamma_{n_{k_2}} \pm \cdots \pm\gamma_{n_{k_s}}$. Thus by [2, Theorem 1.5], $\hat{E}$ is a Sidon set.

Let $H=\{\gamma\in Z^\omega:\gamma=\sum\beta_j\gamma_{n_j},\ \beta_j$ integers, $\sum\beta_j=1\}$. H is a coset of a subgroup of Z^ω and by (d) $\hat{E}=H'\cap P$. By [3, p. 60] there is $\mu\in M(T^\omega)$ such that $\hat{\mu}$ is the characteristic function of H'. μ satisfies condition (b) of the theorem.

3. **Examples.** The corollary shows that there are Bohr sets which are not the finite union of sets with pairwise relatively prime elements. For example, $p_1p_2, p_1p_3, p_4p_5, p_4p_6, p_4p_7, p_8p_9, \cdots$. It is known [3, p. 126] that every infinite subset of a discrete group contains an infinite Sidon subset. However this is not true of Bohr sets.

EXAMPLE. *Let* $F=\{n_j=(p_1p_2\cdots p_j)^j\}$. *Then* F *contains no infinite Bohr subset.*

In fact F contains no infinite subset for which there is a measure satisfying (8). For suppose $E=\{n_{j_1}, n_{j_2},\cdots\}$ and $\hat{\mu}$ satisfies (8). Let μ_k be the translation of μ such that

$$\hat{\mu}_k(\gamma) = \hat{\mu}(\gamma+\gamma_{n_{j_k}}). \tag{10}$$

$\{\mu_k\}$ has a weak star convergent subsequence to a measure $\nu\in M(T^\omega)$ which by a lemma of Helson [3, p. 66] must be singular with respect to Haar measure.

But this is impossible since it is easily seen that

$$\begin{aligned}\hat{\nu}(\gamma) &= 1 \quad \text{if } \gamma = 0,\\ &= 0 \quad \text{if } \gamma \neq 0\end{aligned}$$

so that ν must be the Haar measure.

This example also shows that the corollary is false without (d).

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