

NOTE ON STRUCTURAL STABILITY

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Introduction. In this note we prove that if X is a structural stable vector field on a compact manifold M of dimension $m \geq 2$, then any real valued function defined on M which is invariant under X i.e., which is constant along orbits of X , must be constant. Of course some assumptions of differentiability are made. From this follows in particular that structural stable vector fields do not admit global first integrals. Here we use the ϵ -definition of structural stability, for the non ϵ -definition (see [1]) it is still a problem.

1. We will denote by M a connected compact manifold of dimension $m \geq 2$, with a differentiable structure of class large enough for our purpose. Let $\mathcal{F}_k(M)$ be the set of all real valued functions of class C^k , $k \geq 1$, defined on M and $\mathfrak{X}_k(M)$ the space of all C^k vector fields on M with the C^k -topology. Finally let $\Sigma_k(M)$ be the set of structural stable vector fields contained in $\mathfrak{X}_k(M)$. (See [1].)

Let $X \in \mathfrak{X}_k(M)$. A function $f \in \mathcal{F}_k(M)$ is called *invariant* under X if it is constant along orbits of X . It is equivalent to say that $Xf = 0$, where Xf denotes differentiation of f with respect to the vector field X . An invariant function f under X is called a *global first integral* of X if it is nonconstant on any open subset of M .

LEMMA 1. *If $f \in \mathcal{F}_k(M)$ $k \geq 1$, then the set Γ of critical values of f is closed in R .*

Let $X \in \mathfrak{X}_k(M)$ $k \geq 1$ and $f \in \mathcal{F}_{k+1}(M)$ a nonconstant function invariant under X . We assume that f has at least one regular value and from Lemma 1 it follows then, that there is an open set of them. The inverse image $f^{-1}(a)$ of every regular value a of f is a compact manifold N_a of dimension $(m-1)$ properly imbedded in M . Since f is invariant under X every N_a is invariant under X , i.e., N_a is a union of orbits of X . Let $[a_1, a_2]$ be a closed interval made up of regular values of f , we are going to define a special kind of perturbation of X associated to the interval $[a_1, a_2]$.

Fix a positive definite Riemannian metric g on M of differentiability class large enough. Let $\text{grad } f$ be the vector field on M defined by the equation

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$$Yf = g(Y, \text{grad } f)$$

where Y is any vector field on M . Let h be a real valued function of class C^∞ defined on R such that $h > 0$ on (a_1, a_2) and $h = 0$ otherwise. Then $h \circ f \in \mathfrak{F}_{k+1}(M)$, $h \circ f > 0$ on the open submanifold $P = f^{-1}((a_1, a_2))$ and $h \circ f = 0$ otherwise. Finally for every real number $\alpha > 0$ we define the vector field

$$X_\alpha = X + \alpha(h \circ f)(\text{grad } f).$$

LEMMA 2. X_α for $\alpha > 0$ has the following properties:

1. $X_\alpha \in \mathfrak{F}_k(M)$;
2. $X_\alpha = X$ outside P . In particular N_{a_1} and N_{a_2} are invariant under X_α ;
3. $(X_\alpha)_p \neq 0$ at every point $p \in P$ or in other words X_α has no singularities on P .
4. $(X_\alpha)_p f > 0$ at every point $p \in P$.

Properties 1, 2, and 3 follow straight from the definition of X_α . A direct computation proves Property 4. In fact for any $p \in P$ we have

$$(X_\alpha)_p f = (X + \alpha(h \circ f)(\text{grad } f))_p f = X_p f + \alpha(h \circ f)(p)(\text{grad } f)_p f.$$

But $\alpha > 0$, $(h \circ f)(p) > 0$ and $(\text{grad } f)_p f = g_p(\text{grad } f, \text{grad } f) > 0$, thus $(X_\alpha)_p f > 0$.

LEMMA 3. For every point $p \in P$ the ω -limit set (α -limit set) of p with respect to X_α is contained in $N_{a_2}(N_{a_1})$.

Let ψ_t denote the 1-parameter group of transformations associated with X_α . For every fixed point $p \in P$ we consider the function $f(\psi_t(p))$ as a function of t for $t \geq 0$. From Property 4 of Lemma 2 it follows that $f(\psi_t(p))$ is a strictly increasing function of t . On the other hand $f(\psi_t(p))$ can never reach the value a_2 , because if for some $t_0 > 0$, $f(\psi_{t_0}(p)) = a_2$, then $\psi_{t_0}(p) \in N_{a_2}$ and since N_{a_2} is invariant under X_α , it would imply $p \in N_{a_2}$. We are going to show that $f(\psi_t(p))$ tends to a_2 as t tends to $+\infty$. Suppose $\lim f(\psi_t(p)) = a < a_2$, then by the continuity of f it follows that $\omega(p) \subset N_a$. But this contradicts the fact that $f(\psi_t(q))$ is strictly increasing for every $q \in P$. Thus $\lim f(\psi_t(p))$ must be a_2 . As it was observed above, the continuity of f implies $\omega(p) \subset N_{a_2}$. The proof for the α -limit set is the same.

PROPOSITION. If $X \in \Sigma_k(M)$ $k \geq 1$, then there is no function $f \in \mathfrak{F}_{k+1}(M)$ such that:

- (i) f is invariant under X .
- (ii) f is nonconstant and has at least a regular value.

Suppose there is a function $f \in \mathfrak{F}_{k+1}(M)$ satisfying (i) and (ii). Since f has a regular value, it has a closed interval of them $[a_1, a_2]$. Consider the family X_α of vector fields associated with the interval $[a_1, a_2]$. Let $a = \frac{1}{2}(a_1 + a_2)$ and η the distance between N_a and $N_{a_1} \cup N_{a_2}$. Assume now that X is structural stable, i.e. given $\epsilon > 0$ there exists $\delta > 0$ such that $Y \in \mathfrak{F}_k(M)$ and $d(Y, X) < \delta$ implies there is a ϵ -homeomorphism ϕ of M onto M which sends orbits of X into orbits of Y . Take $\epsilon < \eta/2$ and α such that $d(X_\alpha, X) < \delta$, the corresponding homeomorphism ϕ will move every point of N_a less than ϵ and so the distance between $\phi(N_a)$ and $N_{a_1} \cup N_{a_2}$ will be greater than $\eta/2$. Moreover $\phi(N_a) \subset P$ and is invariant under X_α , but this is a contradiction with Lemma 3.

COROLLARY. *If $X \in \Sigma_k(M)$ $k \geq 1$, then X does not admit a global first integral $f \in \mathfrak{F}_{k+1}(M)$ with a regular value.*

REMARK. If we use Sard's theorem (see [2]) to establish the existence of regular values for a differentiable function f of M into R , then the proposition and the corollary can be set in the following way:

PROPOSITION. *If $X \in \Sigma_m(M)$ where $m = \dim M$, then any function $f \in \mathfrak{F}_{m+1}(M)$ which is invariant under X must be constant.*

COROLLARY. *If $X \in \Sigma_m(M)$ where $m = \dim M$, then X does not admit a global first integral $f \in \mathfrak{F}_{m+1}(M)$.*

BIBLIOGRAPHY

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