WIENER-HOPF TYPE PROBLEMS FOR ELLIPTIC SYSTEMS OF SINGULAR INTEGRAL EQUATIONS[1]

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The problem treated in this paper is roughly the inversion of elliptic systems of singular integral equations in a half-space of R^n. Ellipticity means that the system is invertible over the whole of R^n, in our case explicitly. We first introduce notation and some spaces of (vector-valued) distributions.

Let (x, y) denote points in R^n with $x \in R^{n-1}$, $y \in R$. $R^n_+ [R^n_-]$ is the half-space $y \geqq 0$ $[y \leqq 0]$. $H^{s,p}$ is the space of distributions u for which

$$\|u\|_{s,p} = \|F^{-1}(1 + |\xi|^2 + \eta^2)^{s/2} Fu\|_{L^p} < \infty.$$

Here $(Fu)(\xi, \eta) = \int u(x, y) e^{i(x\cdot\xi + y\cdot\eta)}\, dxdy$, with (ξ, η) dual to (x, y). We assume $1 < p < \infty$. $H^{s,p}_-$ is the subspace of elements supported in R^n_-. $H^{s,p}(R^n_+)$ is the quotient $H^{s,p}/H^{s,p}_-$ (it is a space of distributions on $\mathring{R}^n_+$, the open half-space). Y_+ denotes the canonical map onto the quotient and $\|Y_+ u\|_{s,p}$ is the quotient norm. $H^{s,p}_+$, $H^{s,p}(R^n_-)$ and Y_- are similarly defined. For $s = 0$, we can identify $H^{0,p}_\pm = L^p_\pm$ with $L^p(R^n_\pm)$, and $Y_\pm$ with multiplication by the characteristic function of $R^n_\pm$. The definitions above extend to vector valued functions component-wise.

Let $M(\xi, \eta)$ be an $N \times N$ matrix of functions, positively homogeneous of degree 0, C^{l+1} on $|\xi|^2 + \eta^2 = 1$ where $l > n/2$. The operator $\boldsymbol{M} = F^{-1} M(\xi, \eta) F$ (whose symbol is $M(\xi, \eta)$) is bounded in $H^{s,p}$, invertible (elliptic) if $\det [M(\xi, \eta)] \neq 0$ for $(\xi, \eta) \neq 0$.

THEOREM A. *The operator* $\tilde{\boldsymbol{M}}$: $u \to (Y_- u, Y_+ \boldsymbol{M} u)$ *has a closed range in* $H^{s,p}(R^n_-) \times H^{s,p}(R^n_+)$ *for every s except at most N exceptional values of s*(mod 1). *There exists* $k' \geqq k''$ *such that for* $s = k + \sigma$ *nonexceptional*

$$\|u\|_{s,p} \leqq C[\|Y_- u\|_{s,p} + \|Y_+ \boldsymbol{M} u\|_{s,p}], \quad \text{all } u \in H^{s,p};\ k \geqq k',$$

$$\textstyle\sum_\pm \|V_\pm\|_{-s,p'} \leqq C\|V_- + \boldsymbol{M} V_+\|_{-s,p'}, \quad \text{all } V_\pm \in H^{-s,p'}_\pm\ k \leqq k'';$$

$$k'' = k' \text{ in the scalar case.}$$

The first estimate means that $\tilde{\boldsymbol{M}}$ is 1-1 and has a closed range. The second ("dual") estimate assures that the range of $\tilde{\boldsymbol{M}}$ is full. Thus as $s \to +\infty$ the operator $\tilde{\boldsymbol{M}}$ becomes left invertible, as $s \to -\infty$ it becomes right invertible.

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We proved those estimates before for $n=1$ and general M [6], or $M=I$ (the identity matrix) and general n [7]. In the latter case the exceptional values are $s=1/p$ (mod 1). (This is so whenever $M(0,1)=M(0,-1)$.) In fact, Theorem A shows that $\tilde{M}$ for general M behaves pretty much like $\tilde{I}$.

Theorem A follows from:

THEOREM B. *The operator $Y_+u\to Y_+MY_+u$ has a closed range in $L^p(R^n_+)$ if and only if the eigenvalues of $M(0,-1)\cdot M^{-1}(0,1)$ do not lie on the ray* $\arg\lambda=2\pi/p$. *Let*

$$M_k(\xi,\eta)=(\eta-i|\xi|)^kM(\xi,\eta)(\eta+i|\xi|)^{-k}. \tag{1}$$

There are integers k', k'' such that $Y_+M_sY_+$ is 1-1 if $k\geqq k'$, *onto if $k\leqq k''$ (provided Y_+MY_+ has a closed range).*

REMARK. The estimates of Theorem A settle the problem of obtaining a priori L^p estimates for elliptic partial differential equations in $n+1$ dimensional domains with piecewise smooth boundary operators (so called "mixed" problems). These (usually L^2) estimates were obtained before under very special conditions [3], [4], [5] and [8].

METHOD OF PROOF. For Theorem B, we first prove the following reduction:

LEMMA 1. *Let $M_{\bar{\xi}}=F^{-1}M(\xi,\eta|\xi|^{-1})F$. Then estimate*

$$\|Y_+u\|\leqq C\|Y_+MY_+u\|,\qquad u\in L^p$$

(norms are $L^p(R^n_+)$ norms) is equivalent to the family of estimates

$$\|Y_+u\|\leqq C\|Y_+M_{\bar{\xi}}Y_+u\|,\qquad u\in L^p,$$

for all $\bar{\xi}$ satisfying $|\bar{\xi}|=1$.

The operators $M_{\bar{\xi}}$ are easier to study since their symbols are essentially one-dimensional. As usual with this type of problem, one tries to factor the symbol into product of matrices holomorphic in $\operatorname{Im}\eta>0$ and $\operatorname{Im}\eta<0$. Here it suffices to factor $M(\bar{\xi},\eta)$ for fixed $\bar{\xi}$ and then substitute $\eta|\xi|^{-1}$ for η. We use results of Gohberg-Krein [1], [2]. They factor matrices of the form $I+K(\eta)$ where $K(\eta)\in FL^1$ (or a suitable subring, cf. [1]). Thus we first have to "fill in" the jump of $M(\bar{\xi},\eta)$ at infinity in case $M(0,1)=M(\bar{\xi},\infty)\neq M(\bar{\xi},-\infty)=M(0,-1)$. If the jump matrix

(2) $M(0,-1)\cdot M^{-1}(0,-1)$ is similar to $\operatorname{diag}[\lambda_1,\cdots,\lambda_N]$,

this is readily done by diagonal factors of the form

(3) $$(\eta \pm i)^{\sigma} = \operatorname{diag}\left[(\eta \pm i)^{\sigma}, \cdots, (\eta \pm i)^{\sigma}\right]$$

where $\sigma = (\sigma_1, \cdots, \sigma_N)$ is N-tuple of fractional (may be complex) numbers determined by $\lambda_1, \cdots, \lambda_N$. Indeed $M_1(\xi, \eta) = (\eta - i)^{-\sigma} M(\xi, \eta)(\eta + i)^{\sigma}$ has the same value at $\pm\infty$ and is factorizable. Factoring it, we get for $M(\xi, \eta)$

LEMMA 2. *For a fixed* $\xi \neq 0$ *we have (suppressing the dependence on* ξ*):*

(4) $$M(\xi, \eta) = Q_-^{-1}(\eta)(\eta - i)^{\sigma}\left(\frac{\eta - i}{\eta + i}\right)^{\kappa}(\eta + i)^{-\sigma}Q_+(\eta)$$

where $Q_+(\eta)$ $[Q_-(\eta)]$ *and its inverse are bounded and smooth for real* η, *have holomorphic extension to* $\operatorname{Im}\eta > 0$ $[\operatorname{Im}\eta < 0]$. *Moreover, their derivatives decrease as* $|\eta| \to \infty$ *in a manner which assures that* $Q_\pm(\eta|\xi|^{-1})$ *and their inverses are* L^p*-multipliers.* $\kappa = (\kappa_1, \cdots, \kappa_N)$ *is a nonincreasing sequence of integers which are uniformly bounded for* $|\xi| = 1$.

REMARK. If (2) is not satisfied, the factorization of M is more complicated but the final results remain unchanged.

LEMMA 3. *Let* $\boldsymbol{Q}_\pm = F^{-1}Q_\pm(\eta|\xi|^{-1})F$ *and*

(5) $$\boldsymbol{D}_{\kappa+\sigma} = F^{-1}(\eta - i|\xi|)^{\sigma}\left(\frac{\eta - i|\xi|}{\eta + i|\xi|}\right)^{\sigma}(\eta + i|\xi|)^{-\sigma}F.$$

Then $\boldsymbol{Q}_+$ *sets an isomorphism between the null-spaces of* $Y_+\boldsymbol{M}_\xi Y_+$ *and* $Y_+\boldsymbol{D}_{\kappa+\sigma}Y_+$. $\boldsymbol{Q}_-$ *sets an isomorphism between the ranges of these operators in* $L^p(R^n_+)$.

Notice that $\boldsymbol{D}_{\kappa+\sigma}$ is a direct sum of scalar operators for which we have

LEMMA 4. *If* k *is an integer and* $-1 + 1/p < \operatorname{Re}\sigma \leq 1/p$ *then the (scalar) operator* $Y_+\boldsymbol{D}_{\kappa+\sigma}Y_+$ *has a closed range if and only if* $\operatorname{Re}\sigma \neq 1/p$. *In this case it is* 1-1 *if* $k \geq 0$, *onto if* $k \leq 0$.

The proof of Theorem B follows now quite easily from Lemmata 1–4; for Theorem A we need two more. Let us denote

$$J_\pm^t = F^{-1}\left[\eta \pm i(1 + |\xi|^2)^{1/2}\right]^t F.$$

LEMMA 5. J_-^t *maps* $H^{s,p}(R^n_+)$ *onto* $H^{s-t,p}(R^n_+)$ *(in particular onto* $L^p(R^n_+)$ *if* $s = t$*) and* $\|Y_+J_-^t u\|_{s-t,p} \sim \|Y_+u\|_s$. *(Similar statement with* $+$ *and* $-$ *interchanged.)*

LEMMA 6. *The following estimates are equivalent*

$$\|u\|_{s,p} \leqq C[\|Y_-u\|_{s,p} + \|Y_+Mu\|_{s,p}], \qquad u \in H^{s,p},$$
$$\|v\|_{0,p} \leqq C[\|Y_-v\|_{0,p} + \|Y_+\overset{s}{J_-}M\overset{-s}{J_+}v\|_{0,p}], \qquad v \in L^p,$$
$$\|Y_+v\|_{0,p} \leqq C\|Y_+M_sY_+v\|, \qquad v \in L^p \qquad \text{(cf. (1))},$$

$$\textstyle\sum_{\pm} \|V_\pm\|_{-s,p'} \leqq C\|V_- + MV_+\|_{-s,p'}, \; V_\pm \in H^{-s,p'}.$$

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