ON NONLINEAR ELLIPTIC BOUNDARY-VALUE PROBLEMS

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Communicated by F. Browder, October 26, 1965

The purpose of this note is to prove the solvability of a nonlinear elliptic equation with general boundary conditions. Nonlinear variational elliptic boundary-value problems have been considered by Browder in [**4**], [**5**] and by Visik.

In §1, we give the notations. In §2, we prove the solvability of the nonlinear elliptic equation with linear boundary conditions and in §3, we consider the case when we have a nonlinear boundary condition.

The writer is indebted to Professor Felix Browder for his criticisms and suggestions.

1. Let G be a bounded, open subset of E^n with a C^∞ imbedding mapping of its boundary Γ into E^n. Let A be a linear elliptic differential operator of order $2m$ with coefficients defined on G; and $a(x, \xi)$ its characteristic form. Let $B_1, \cdots, B_m$ be m linear differential operators of orders r_j with coefficients defined on Γ and let $b_j(x, \xi)$ be their characteristic forms.

We set:

$$D_j = i^{-1}\partial/\partial x_j; \quad j = 1, \cdots, n,$$

$$D^\alpha = \prod_{j=1}^{n} D_j^{\alpha_j}; \qquad |\alpha| = \sum_{j=1}^{n} \alpha_j, \quad \alpha_j \geqq 0.$$

The elliptic boundary-problem $\{A; B_j; j=1, \cdots, m\}$ on G is assumed to be uniformly regular in the sense of Browder [**3**].

We now state our main assumption on $\{A; B_j\}$:

ASSUMPTION 1. *Let $\{A; B_j; j=1, \cdots, m\}$ be a uniformly regularly elliptic boundary problem on G. We assume that*:

(i) $a(x, \xi)/|a(x, \xi)| \neq -1$ *for x in G, ξ in R^n.*

(ii) *if* $c_{rj}(x, T, t) = \int_C \lambda^{r-1} b_j(x, \lambda N_x + T)[a(x, \lambda N_x + T) + t]^{-1} d\lambda$ *where C is a closed, Jordan rectifiable curve in the λ upper half plane containing all the m roots of $a(x, \lambda N_x + T) + t$ and N_x is the unit outer normal to Γ at x; T is any tangent vector to Γ at x; then there exists a positive constant c independent of x, t such that*:

$$|\operatorname{Det}(c_{rj}(x, T, t))| \geqq c \quad \textit{for } t \geqq t_0 > 0$$

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and for all unit tangent vectors T to Γ at x. (Cf. Agmon [1], *Visik-Agranovich* [7].)

We define for $1<p<\infty$:

$$W^{2m,p}(G) = \{u: u \text{ in } L^p(G),\ D^\alpha u \text{ in } L^p(G) \text{ for } |\alpha| \leqq 2m\}.$$

It is a separable, reflexive Banach space with the norm:

$$\|u\|_{2m,p} = \left\{ \sum_{|\alpha|\leq 2m} \|D^\alpha u\|^p_{L^p(G)} \right\}^{1/p}.$$

For $k\geqq 0$, we define $W^{k,2}(\Gamma)$ as the completion of $C^\infty(\Gamma)$ with respect to the norm:

$$\|u\|_{W^{k,2}(\Gamma)} = \left\{ \sum_{j=1}^{N} \|\phi_j u\|^2_{W^{k,2}(E^{n-1})} \right\}^{1/2} \quad u \text{ in } C^\infty(\Gamma)$$

where $\{\phi_j\}$ is a finite partition of unity corresponding to a covering of G; and $\|\phi_j u\|_{W^{k,2}(E^{n-1})}$ is taken in local coordinates, in the usual fashion. One can show that the norm does not depend on the ϕ_j.

2. The main result of this section is the following theorem:

THEOREM 2.1. *Let $\{A; B_j; j=1, \cdots, m\}$ be a uniformly regularly elliptic boundary problem on G and satisfying Assumption* 1. *The linear differential operators A, B_j are of orders $2m$, r_j respectively with $r_j<2m$ and with coefficients of class C^{2m} on $G\cup\Gamma$. Let $f(x, \zeta_1, \cdots, \zeta_{2m-1})$ be a function, measurable in x on G and continuous in $(\zeta_1, \cdots, \zeta_{2m-1})$. Suppose that:*

$$|f(x, u, \cdots, D^{2m-1}u)| \leqq M\left\{1 + \sum_{|\alpha|\leq 2m-1} |D^\alpha u|\right\}$$

where M is a constant. Then the nonlinear elliptic boundary problem:

$$(A + tI)u = f(x, u, \cdots, D^{2m-1}u) \quad \text{on } G;$$
$$B_j u = g_j(x) \quad \text{on } \Gamma, \qquad j = 1, \cdots, m$$

has a nontrivial solution u in $W^{2m,2}(G)$ for $t\geqq t_0 > 0$ if $(f(x, 0, \cdots, 0), g_1, \cdots, g_m)$ is a nonzero vector of $L^2(G)\times\prod_{j=1}^m W^{2m-r_j-1/2,2}(\Gamma)$.

PROOF. If v is in $W^{2m,2}(G)$, then $f(x, \lambda v, \cdots, \lambda D^{2m-1}v)$ where $0\leqq\lambda\leqq 1$, belongs to $L^2(G)$. From Visik-Agranovich's result (Theorem 5.1) [7], there exists a unique solution u in $W^{2m,2}(G)$ of the linear boundary problem:

$(A+tI)u=f(x, \lambda v, \cdots, \lambda D^{2m-1}v)$ on G; $B_ju=g_j$ on Γ, $j=1, \cdots, m$.

Set $T(\lambda)v=u$. $T(\lambda)$ is a nonlinear mapping of $W^{2m,2}(G)$ into itself and is defined on all of $W^{2m,2}(G)$.

PROPOSITION 2.1. *$T(\lambda)v$ is completely continuous in $[\lambda, v]$ from $W^{2m,2}(G)\times[0,1]$ to $W^{2m,2}(G)$.*

PROOF. Suppose that $v_n\to v$ in $W^{2m,2}(G)$, $\lambda_n\to\lambda$. Let $u_n=T(\lambda_n)v_n$ and $u=T(\lambda)v$. Then we have the following estimate:

$$\|u_n-u\|_{2m,2}\leqq C\|f(x,\lambda_n v_n,\cdots,\lambda_n D^{2m-1}v_n)-f(x,\lambda v,\cdots\lambda D^{2m-1}v)\|_{0,2}$$

C is a constant independent of u_n, u, f, λ, λ_n. (Theorem 4.1 of [7]) $\lambda_n D^\alpha v_n$ converges in measure to $\lambda D^\alpha v$, for $|\alpha|\leqq 2m$. An argument as given by Browder in [5, Lemma 3.1] shows that: $f(x,\lambda_n v_n,\cdots,\lambda_n D^{2m-1}v_n)$ converges in measure to $f(x,\lambda v,\cdots,\lambda D^{2m-1}v)$. Then as in Lemma 3.2 of [5], the mapping: $F(\lambda)w=f(x,\lambda w,\cdots,\lambda D^{2m-1}w)$ of $\prod_{j=1}^{2m-1}L^2(G)\times[0,1]$ into $L^2(G)$ is continuous so that $u_n\to u$ in $W^{2m,2}(G)$. To show that $T(\lambda)$ is completely continuous it suffices to prove that $T(\lambda)$ is compact. Let $\|v_n\|_{2m,2}\leqq M$, then from the Sobolev Imbedding Theorem, there exists a subsequence $\{v_n\}$ which converges to v in $W^{2m-1,2}(G)$. From (i), $u_n\to u$ in $W^{2m,2}(G)$, so that $\{u_n\}$ is a compact set of $W^{2m,2}(G)$.

PROPOSITION 2.2. *The operator $I-T(0)$ is a homeomorphism of $W^{2m,2}(G)$. If u is a solution of $u=T(\lambda)u$, then $\|u\|_{2m,2}\leqq M$ where M is a constant independent of λ.*

PROOF. Let u_0 be the unique solution of: $(A+tI)u_0=f(x,0,\cdots,0)$ on G and $B_ju_0=g_j$ on Γ, $j=1,\cdots,m$. Then $T(0)v=u_0$ for all v in $W^{2m,2}(G)$. It is now obvious that $I-T(0)$ is a homeomorphism of $W^{2m,2}(G)$.

From Visik-Agranovich's result (Theorem 4.1 of [7]), we have the following a priori estimate for u_0:

$$\|u_0\|_{W^{2m,2}(G)}+t^{1/2m}\|u_0\|_{W^{2m-1,2}(G)}\leqq C\Big\{\|f(x,0,\cdots,0)\|_{L^2(G)} + \sum_{j=1}^{m}\|g_j\|_{W^{2m-r_j-1/2,2}(\Gamma)}+t^{1-(r_j+1/2)/2m}\|g_j\|_{L^2(\Gamma)}\Big\}.$$

Let w be the solution of:

$$(A+tI)w=f(x,\lambda u,\cdots,\lambda D^{2m-1}u)-f(x,0,\cdots,0)\quad\text{on } G;$$
$$B_jw=0\quad\text{on }\Gamma;\qquad j=1,\cdots,m.$$

Then:

$$\|w\|_{2m,2}+t^{1/2m}\|w\|_{2m-1,2}\leqq C\{M\|u\|_{2m-1,2}+\|f(x,0,\cdots,0)\|_{0,2}\}.$$

C is a constant independent of $t, w, u, f(x, 0, \cdots, 0), \lambda$.

On the other hand, if u is a solution of $u = T(\lambda)u$, then $u = u_0 + w$ for $(A + tI)(u - u_0 - w) = 0$ on G and $B_j(u - u_0 - w) = 0$ on Γ, $j = 1, \cdots, m$

Hence: $\|u\|_{2m,2} + t^{1/2m}\|u\|_{2m-1,2}$ is majorized by:

$$C\left\{M\|u\|_{W^{2m-1,2}(G)} + \|f(x, 0, \cdots, 0)\|_{L^2(G)} + \sum_{j=1}^{m} \|g_j\|_{W^{2m-r_j-1/2,2}(\Gamma)} + t^{1-(r_j+1/2)/2m}\|g_j\|_{L^2(\Gamma)}\right\}.$$

We take t such that $2CM \leqq t^{1/2m}$, then: $\|u\|_{2m,2} \leqq M$. The proposition is proved.

We return to the proof of the theorem.

The operator $T(\lambda)$ satisfies all the conditions of the Leray-Schauder Theorem [6].[2] Hence $T(1)$ has a fixed point; i.e. $T(1)u = u$ and so the nonlinear elliptic boundary problem considered has a solution u in $W^{2m,2}(G)$. Since $(f(x, 0, \cdots, 0), g_1, \cdots, g_m)$ is a nonzero vector ,u is nontrivial. The theorem is proved.

With a stronger hypothesis on $f(x, \zeta_1, \cdots, \zeta_{2m-1})$, we also have unicity

THEOREM 2.2. *With the hypotheses of Theorem 2.1, suppose that $f(x, \zeta_1, \cdots, \zeta_{2m-1})$ satisfies a Lipschitz condition with respect to $\zeta_1, \cdots, \zeta_{2m-1}$. Then the nonlinear elliptic boundary problem:*

$$(A + tI)u = f(x, u, \cdots, D^{2m-1}u) \quad \text{on } G$$

and

$$B_j u = g_j(x) \quad \text{on } \Gamma, \qquad j = 1, \cdots, m$$

has a unique solution u in $W^{2m,2}(G)$ for $t \geqq t_0 > 0$.

PROOF. From Theorem 2.1, the above nonlinear problem has a solution. Suppose that u, u_0 are two solutions with $u \neq u_0$.

Let v be the solution of:

$$(A + tI)v = f(x, u, \cdots, D^{2m-1}u) - f(x, u_0, \cdots, D^{2m-1}u_0) \quad \text{on } G$$

and

$$B_j v = 0 \quad \text{on } \Gamma, \qquad j = 1, \cdots, m.$$

From [7], it follows that v is unique and the following estimate holds:

[2] The uniform continuity condition in the Leray-Schauder theorem is not necessary as observed by Browder in *Problèmes non-linéaires*, Séminaire Math. Sup. Été., University of Montreal, 1965.

$$\|v\|_{2m,2} + t^{1/2m}\|v\|_{2m-1,2} \leqq C\|f(x, u, \cdots, D^{2m-1}u) - f(x, u_0, \cdots, D^{2m-1}u_0)\|_{0,2}.$$

C is a constant independent of v, u, u_0, t.

Since $f(x, \zeta_1, \cdots, \zeta_{2m-1})$ satisfies a Lipschitz condition with respect to $\zeta_1, \cdots, \zeta_{2m-1}$; we get:

$$t^{1/2m}\|v\|_{2m-1,2} \leqq C\|u - u_0\|_{2m-1,2}.$$

On the other hand, $v = u - u_0$ for $(A + tI)(v - u + u_0) = 0$ on G and $B_j(v - u + u_0) = 0$ on Γ, $j = 1, \cdots, m$. (Cf. Theorem 5.1 of [7].) Hence: $t^{1/2m}\|u - u_0\|_{2m-1,2} \leqq C\|u - u_0\|_{2m-1,2}$.

Take t such that $Ct^{-1/2m} < 1$ and we get a contradiction. Therefore $u = u_0$.

THEOREM 2.3. (i) *With the hypotheses of Theorem* 2.1, *the nonlinear elliptic boundary problem*: $(A + tI)u = f(x, u, \cdots, D^{2m-1}u)$ *on* G; $B_j u = 0$ *on* Γ; $j = 1, \cdots, m$ *has a solution* u *in* $W^{2m,p}(G)$ *for* $t \geqq t_0 > 0$, $1 < p < \infty$.

(ii) *Suppose that*:

$$|f(x, u, \cdots, D^k u)| \leqq M\left\{1 + \sum_{|\alpha| \leq k} |D^\alpha u|^{p-1}\right\}$$

with $k \leqq 2m - 1 - n/p + n/p(p-1)$, $1 < p < \infty$. *M is a constant. Then*: $(A + tI)u = f(x, u, \cdots, D^k u)$ *on* G; $B_j u = 0$ *on* Γ, $j = 1, \cdots, m$ *has a nontrivial solution* u *in* $W^{2m,p}(G)$ *if* $f(x, 0, \cdots, 0) \neq 0$.

PROOF. (i) The proof is the same as that of Theorem 2.1 with the exception that instead of using the a priori estimate established by Visik-Agranovich in [7], we use the following inequality obtained by Agmon in [1] for the case $g_j = 0$; $j = 1, \cdots, m$

$$\sum_{k=0}^{2m} t^{1-k/2m}\|u\|_{k,p} \leqq C\|(A + tI)u\|_{0,p}.$$

(ii) The second part of the theorem follows as above by observing that from the Sobolev Imbedding Theorem, $D^\alpha v$ belongs to $L^{p(p-1)}(G)$ for $|\alpha| \leqq k$, v in $W^{2m,p}(G)$ if $k \leqq 2m - 1 - n/p + n/p(p-1)$. Moreover the mapping F from $\prod_{j=1}^{2m-1} L^{p(p-1)}(G) \times [0, 1]$ into $L^p(G)$ is continuous (cf. Browder [5, Lemma 3.2]) and: $\|F(\lambda)u\|_{0,p} \leqq M\{1 + \|u\|_{2m-1,p}\}$.

3. In this section we consider the case when we have a nonlinear boundary condition.

THEOREM 3.1. *Let* $\{A; B_j; j = 1, \cdots, m\}$ *be a uniformly regularly* *elliptic boundary problem on G and satisfying Assumption 1. The linear differential operators A, B_j are of orders $2m$, r_j respectively with $r_j<2m$ and coefficients of class C^{2m} on $G\cup\Gamma$. Suppose that $r_m=2m-1$. Let $f(x, \zeta_1, \cdots, \zeta_{2m-1})$ be as in Theorem 2.1 and $g_m(x, \zeta)$ be a function twice continuously differentiable with respect to x, ζ and such that:*

$$\sum_{|\alpha|\leqq 2}\left|\frac{\partial^\alpha}{\partial^{\alpha_1}x\partial^{\alpha_2}(D^\beta v)}\, g_m(x, D^\beta v)\right| \leqq M$$

$|\beta|\leqq 2m-3$. M is a constant.

Then the nonlinear elliptic boundary problem:

$$\begin{aligned}(A+tI)u &= f(x, u, \cdots, D^{2m-1}u) \quad \text{on } G,\\ B_j u &= g_j(x) \quad \text{on } \Gamma; \qquad j=1,\cdots,m-1,\\ B_m u &= g_m(x, D^\beta u) \quad \text{on } \Gamma,\end{aligned}$$

has a nontrivial solution u in $W^{2m,2}(G)$ for $t\geqq t_0>0$ if $(f(x, 0, \cdots, 0), g_1(x), \cdots, g_{m-1}(x), g_m(x, 0))$ is a nonzero vector of

$$L^2(G)\times\prod_{j=1}^{m} W^{2m-r_j-1/2,2}(\Gamma).$$

PROOF. With the above assumption on $g_m(x, \zeta)$; it follows that $g_m(x, D^\beta v)$ belongs to $W^{2,2}(G)$ if v is in $W^{2m,2}(G)$.

Let v be an element of $W^{2m,2}(G)$. In §2 we define the nonlinear operator $T(\lambda)$ acting from $W^{2m,2}(G)$ into itself as follows:

$$T(\lambda)v = u, \qquad 0\leqq\lambda\leqq 1,$$

where u is the unique solution of the linear elliptic boundary problem:

$$\begin{aligned}(A+tI)u &= f(x, \lambda v, \cdots, \lambda D^{2m-1}v) \quad \text{on } G,\\ B_j u &= g_j(x) \quad \text{on } \Gamma; \qquad j=1,\cdots,m-1,\\ B_m u &= g_m(x, \lambda D^\beta v) \quad \text{on } \Gamma; \qquad |\beta|\leqq 2m-3.\end{aligned}$$

To prove the theorem, it suffices to show that the operator $T(\lambda)$ satisfies all the conditions of the Leray-Schauder Theorem and so has a fixed point. An elementary computation shows that: $Dg_m(x, \lambda D^\beta v)$; $D^2 g_m(x, \lambda D^\beta v)$, $g_m(x, \lambda D^\beta v)$ are continuous functions of x, $\lambda D^\beta v$, $\lambda D^{\beta+1}v$, $\lambda D^{\beta+2}v$ and moreover:

$$\sum_{|\alpha|\leqq 2} |D^\alpha g_m(x, \lambda D^\beta v)| \leqq M\left\{1+\sum_{|\alpha|\leqq 2m-1}|D^\alpha v|\right\}.$$

With the above remark, the proofs of Propositions 2.1 and 2.2 may be carried over immediately with some obvious changes. The theorem is proved.

REMARK. It was pointed out to the writer by Professor Felix Browder that the results of this note can be obtained by a simpler argument using the Schauder fixed point theorem rather than the Leray-Schauder Theorem.

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