RESEARCH PROBLEMS

3. Richard Bellman: *Singular differential and the Lagrange expansion.*

Consider the differential equation $u=g(t)+\epsilon h(u, du/dt)$, where h is analytic in its arguments, and suppose that we iterate, $u=g+\epsilon h(g, g')+\epsilon^2 h_1+\cdots$, a formal power series in ϵ. Is there an extension of the Lagrange expansion theorem which permits us to obtain the coefficients in a reasonably simple fashion? (Received October 4, 1965.)

4. Richard Bellman: *Lagrange expansion for functionals.*

The partial differential equation $u_t=uu_x$, $u(x, 0)=g(x)$, which has the solution $u=g(x+tu)$ can be used to derive the Lagrange expansion of the solution of $v=c+\epsilon h(v)$ as a power series in ϵ. Can one find an analogous solution of the functional partial differential equation $u_t=u(\delta u/\delta f)$, $u(f, 0)=g(f)$, where $g(f)$ is a given functional of f, and $\delta u/\delta f$ denotes the Gateaux derivative of the functional $u(f, t)$, which can be used to find a Lagrange expansion of the solution of $v=c(x)+\epsilon h(v)$, where h is a functional of v? (Received October 4, 1965.)

5. J. M. Gandhi: *The number of representations of a number as a sum of ten squares.*

Let $\gamma_{2s}(n)$ denote the number of representations of a number n as a sum of $2s$ squares. We shall not discuss representations of n as a sum of an odd number of squares. As usual we call a function $f(n)$ multiplicative if it satisfies the condition

$$f(mn) = f(m)f(n), \qquad (m, n) = 1.$$

It is observed that $\gamma_{2s}(n)$ is not multiplicative for all values of $2s$. For example $\gamma_6(n)$, $\gamma_{24}(n)$, etc. are not multiplicative. But $\gamma_{2s}(n)$ is multiplicative for some values of s. For example it was shown by Gupta and Vaidhya [2] that $\gamma_2(n)/4$ is multiplicative. Since $\sigma(n)$, the sum of the divisors of n and $\sigma_0(n)$, sum of the odd divisors of n are multiplicative and

$$\gamma_4(2n+1) = 8\sigma(2n+1),$$
$$\gamma_4(2n) = 24\sigma_0(n),$$

[3, p. 132]

hence $\gamma_4(2n+1)/8$ and $\gamma_4(2n)/24$ are multiplicative. Also

$$\gamma_8(n) = (-1)^{n-1}16\Sigma_d(-1)^{d-1}d^3$$

[1, p. 315].

But $f(d) = \Sigma(-1)^{d-1}d^3$ is multiplicative [4, p. 77] and hence $\gamma_8(n)/16(-1)^{n-1}$ is also multiplicative.

Thus the nature of γ_{2s} for $s \leqq 4$ is known. I have observed that $\gamma_{10}(n)$ is multiplicative but could not prove it. Hence we propose the following

PROBLEM. Prove that $\gamma_{10}(n)$ is multiplicative. It may be noted that $\gamma_{10}(n)$ can be expressed as [1, p. 315] or [3, p. 133]

$$\gamma_{10}(n) = \tfrac{4}{5}[E_4(n) + 16E_4'(n) + 8x_4(n)]$$

where

$$E_4(n) = \sum_{d \text{ odd}, d|n} (-1)^{(d-1)/2}d^4,$$

$$E_4'(n) = \sum_{d \text{ odd}, d|n} (-1)^{(d'-1)/2}d^4,$$

(d' being $n|d$, the divisor of n "conjugate" to d) and

$$x_4(n) = \frac{1}{4} \sum_{a^2+b^2=n} (a + b_i)^4, \text{ where } i = (-1)^{1/2}.$$

It is likely that there may exist other values of s for which $\gamma_{2s}(n)$ is multiplicative.

REFERENCES

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2. H. Gupta and A. M. Vaidhya, *The number of representations of a number as a sum of two squares*, Amer. Math. Monthly 70 (1963), 1081.

3. G. H. Hardy, *Ramanujan*, Chelsea, New York, 1940.

4. J. V. Uspensky and M. A. Heaslet, *Elementary number theory*, McGraw-Hill, New York, 1939.

(Received October 15, 1965.)

6. R. M. Redheffer: *Three problems in elementary analysis.*

A set of complex exponentials $\{e^{i\lambda_n x}\}$ is said to have completeness interval I if the equations

$$\int_a^b e^{i\lambda_n x} f(x)\, dx = 0, \qquad n = 0, \pm 1, \pm 2, \cdots$$

possess a nontrivial solution $f \in L^p$ when $b-a>I$, and no such solution when $b-a<I$. We put $I=0$ if the set is incomplete on every interval, and $I=\infty$ if it is complete on every finite interval. Clearly I is independent of (a, p).

PROBLEM A. Let $\{\lambda_n\}$ and $\{\mu_n\}$ be sequences of complex numbers such that

$$\sum \left| \frac{1}{\lambda_n} - \frac{1}{\mu_n} \right| < \infty.$$

Do $\{e^{i\lambda_n x}\}$ and $\{e^{i\mu_n x}\}$ have the same completeness interval?

PROBLEM B. If $F(z)$ is an even entire function of order less than 2, let λ_n be the roots of $F(z)=a$ and μ_n the roots of $F(z)=b$. Do $\{e^{i\lambda_n x}\}$ and $\{e^{i\mu_n x}\}$ have the same completeness interval?

PROBLEM C. Let C be the class of even entire functions, of order less than 2, with real zeros only, whose counting-function Λ satisfies

$$\int_1^\infty \frac{|\Lambda(u) - u|}{u^2}\, du < \infty.$$

Let $\{x_n\}$ be a sequence of positive numbers with $\inf x_{n+1}/x_n > 1$, and let $\{y_n\}$ be a sequence of real numbers. By elementary methods one can show that the equations $F(x_n) = y_n$ have a solution $F \in C$ if and only if

$$\sum \left(\frac{\log^+ |y_n|}{x_n} \right)^2 < \infty.$$

Allowing complex zeros of F, obtain a corresponding result for complex x_n and y_n. (Received October 15, 1965.)

7. Srisakdi Charmonman: *Eigenvalues of a $2n \times 2n$ matrix.*

If A_n and B_n are real square matrices of order n, and

$$A_{2n} = \begin{bmatrix} A_n & B_n \\ B_n & A_n \end{bmatrix}$$

then

$$|A_{2n} - \lambda I_{2n}| = |X_n - \lambda I_n| \, |Y_n - \lambda I_n|$$

where $X_n = A_n + B_n$ and $Y_n = A_n - B_n$. In other words, the task of solving for eigenvalues of a $2n \times 2n$ matrix can be reduced to solving for eigenvalues of two $n \times n$ matrices.

1. Can we give explicit formulae for the two $n \times n$ matrices when

$$A_{2n} = \begin{bmatrix} A_n & B_n \\ B_n^T & A_n \end{bmatrix}?$$

2. If

$$A_{4n} = \begin{bmatrix} A_{2n} & B_{2n} \\ B_{2n} & A_{2n} \end{bmatrix}$$

it is conceivable that we can write

$$| A_{4n} - \lambda I_{4n} | = | P_n - \lambda I_n | | Q_n - \lambda I_n | | R_n - \lambda I_n | | S_n - \lambda I_n | .$$

Can we give explicit formulae for the four $n \times n$ matrices in terms of X_{2n} and Y_{2n}? (Received November 8, 1965.)

8. Ralph DeMarr: *Complete lattices and compact Hausdorff spaces.*

CONJECTURE. Every compact Hausdorff space can be partially ordered so that it becomes a complete lattice in which order (o-convergence) of arbitrary nets coincides with the topological convergence (for definitions see [**1**, p. 59] and [**3**, p. 65]). The problem is to determine whether this conjecture is true or false. The conjecture is true for the circle, for example; simply take the set $\{(x, 0) : 0 \leqq x < 1\} \cup \{(x, 1) : 0 < x \leqq 1\}$ in the plane with the usual coordinatewise partial ordering; i.e., $(x_1, y_1) \leqq (x_2, y_2)$ if and only if $x_1 \leqq x_2$ and $y_1 \leqq y_2$.

Even if the above conjecture is not true in general, in those cases where it is true it would still be interesting to know what contribution the lattice structure makes to our understanding of the topological structure.

REFERENCES

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2. R. E. DeMarr, *Order convergence and topological convergence*, Proc. Amer. Math. Soc. **16** (1965), 588–590.

3. J. L. Kelley, *General topology*, Van Nostrand, New York, 1955.

(Received November 10, 1965.)

9. Fred Gross: *Analytic functions.*

One can show easily enough that if $\{f_\alpha; \alpha \in A\}$ is a nondenumerable family of functions analytic on a region D, then there exists a point $z_0 \in D$ such that the set $S(z_0) = \{f_\alpha(z_0)\}$ has a limit point.

Must there exist a point $z_1 \in D$ such that $S(z_1)$ is nondenumerable? (Received November 30, 1965.)