SMOOTHING LOCALLY FLAT IMBEDDINGS[1]

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The fundamental imbedding problem for manifolds is to classify the imbeddings of an n-manifold into a q-manifold under ambient isotopy. We announce here that the differentiable and topological cases of this problem for differentiable manifolds are the same if $2q>3(n+1)$ and $q\geqq 8$.

This follows from Theorem 2 below which states that a locally flat imbedding of a compact differentiable manifold M^n into a differentiable manifold Q^q is ambient isotopic to a differentiable imbedding if $2q>3(n+1)$ and $q\geqq 8$. Since this ambient isotopy may be chosen arbitrarily close to the identity map, the set of differentiable imbeddings is dense in the set of locally flat imbeddings of M^n in Q^q.

It will then follow that two locally flat imbeddings of M^n into Q^q are ambient isotopic if they are homotopic; hence the classification problem reduces to a problem in homotopy theory.

THEOREM 1. *Let $f\colon B^n\rightarrow \operatorname{int} Q^q$ be a locally flat imbedding of the unit n-ball into Q^q. Such an f always extends to $f\colon R^q\rightarrow \operatorname{int} Q^q$. Let C^{n-1} be a compact differentiable submanifold of $\partial B^n=S^{n-1}$, and suppose that f is differentiable on a neighborhood of C^{n-1} in B^n. Let $q\geqq 7$, $2q>3(n+1)$ and $\epsilon>0$. Then there exists an ambient ϵ-isotopy $F_t\colon Q^q\rightarrow Q^q$, $t\in[0,1]$, satisfying*

(1) $F_0=$ identity,

(2) F_1f is differentiable on $\operatorname{int} B^n$ *and on a neighborhood of C^{n-1} in B^n,*

(3) $F_t=$ identity on $Q-N_\epsilon(f(B^n))$ and on $f(R^n-\operatorname{int} B^n)$ for all $t\in[0,1]$,

(4) $|F_t(x)-x|<\epsilon$ for all $x\in Q^q$ and $t\in[0,1]$. ($N_\epsilon(X))$ is the set of points within ϵ of X.)

THEOREM 2. *Let $f\colon M^n\rightarrow Q^q$ be a locally flat imbedding such that either $f(M^n)\subset \operatorname{int} Q^q$ and $q\geqq 7$ or $f^{-1}(\partial Q^q)=\partial M^n$ and $q\geqq 8$. Let $2q>3(n+1)$ and $\epsilon>0$. Then there exists an ambient ϵ-isotopy $F_t\colon Q^q\rightarrow Q^q$, $t\in[0,1]$, satisfying*

(1) $F_0=$ identity,

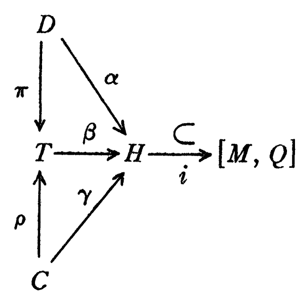
(2) F_1f is a differentiable imbedding,

[1] This is an announcement of a portion of the author's dissertation at the University of Chicago written under Professor Eldon Dyer.

(3) $F_t = identity$ *on* $Q - N_\epsilon(f(M^n))$ *for all* $t \in [0, 1]$,
(4) $|F_t(x) - x| < \epsilon$ *for all* $x \in Q^q$ *and* $t \in [0, 1]$.

The proof follows from Theorem 1 by considering the handlebody decomposition of M^n, and smoothing the imbedding of one handle at a time.

Only imbeddings of M^n into Q^q satisfying $f(M^n) \subset \text{int}\, Q^q$ or $f^{-1}(\partial Q^q) = \partial M^n$ will be considered. Let T be the set of equivalence classes of locally flat imbeddings of M^n into Q^q under equivalence by ambient isotopy. Similarly, let $D(C)$ be the set of equivalence classes of differentiable (combinatorial) imbeddings of M^n into Q^q under equivalence by ambient diffeotopy (ambient combinatorial isotopy). Let H be the homotopy classes of locally flat imbeddings of M^n into Q^q. H is a subset of $[M^n, Q^q]$, the homotopy classes of maps of M^n into Q^q. Then we have the following commutative diagram where the maps are the natural projections.



β is clearly onto for all n and q. Gluck has shown [1] that ρ and γ, and hence β and βi are isomorphisms for $q \geqq 2n+2$. Haefliger has shown [2] that π is a monomorphism and that α is an isomorphism if $2q > 3(n+1)$.

It follows from Theorem 2 that π is also epimorphic if $2q > 3(n+1)$ and either $q \geqq 7$ when $f(M^n) \subset \text{int}\, Q^q$ or $q \geqq 8$ when $f^{-1}(\partial Q^q) = \partial M^n$. Then π and β are isomorphisms in this range of dimensions.

References

1. H. Gluck, *Embeddings in the trivial range*, Ann. of Math. **81** (1965), 195–210.

2. A. Haefliger, *Plongements différentiables de variétés dans variétés*, Comment. Math. Helv. **36** (1961), 47–82.

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