PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS WITH UNIFORMLY CONTINUOUS COEFFICIENTS

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Consider the following system of equations:

$$(1)\qquad \sum_{j=1}^{M}\sum_{|\alpha|\leq m} a_\alpha^{k,j}(x,t)(\partial/\partial x)^\alpha u_j - \delta_{kj}(\partial/\partial t)u_k = f_k, \qquad k=1,\cdots,M$$

here x is a point in E^n and $t\in(0,R)$, $R<\infty$.

Assume the system (1) is parabolic in the sense of I. G. Petrovsky, i.e. the roots $\lambda(x,t;z)$ of the equation:

$$\mathrm{Det}\left(\sum_{|\alpha|=m} a_\alpha^{k,j}(x,t)(iz)^\alpha - \delta_{kj}\lambda\right)=0$$

satisfy $\mathrm{Re}(\lambda(x,t;z))<-\delta<0$ for $|z|=1$, independent of (x,t).

Define $L_0^{p,m,1}(E^n x(0,R))$ to be the closure in the class of distributions of E^{n+1}, of the functions $u\in C_0^\infty(E^n x(0,\infty))$ with respect to the norm:

$$\|u\|_{m,1} = \sum_{|\alpha|\leq m}\left(\int_0^R\int_{E^n} |(\partial/\partial x)^\alpha u|^p\,dxdt\right)^{1/p} + \left(\int_0^R\int_{E^n} |\partial/\partial t|^p\,dxdt\right)^{1/p}.$$

Define $(L_0^{p,m,1})^M$ to be all vectors $u=(u_1,\cdots,u_M)$ with $u_k\in L_0^{p,m,1}(E^n x(0,R))$.

Concerning the coefficients of (1), assume that:

(i) $a_\alpha^{k,j}(x,t)$ are bounded and measurable over $E^n x(0,R)$ for all α,k,j,

(ii) for $|\alpha|=m$, $a_\alpha^{k,j}(x,t)$ are uniformly continuous in $E^n x(0,R)$, for all k,j.

THEOREM. *Given any vector-valued* $f=(f_1,\cdots,f_M)$, *where* $f_k\in L^p(E^n x(0,R))$, *there exists a unique* $u\in(L_0^{p,m,1})^M$ *satisfying system* (1).

The proof of this theorem is based upon the following representation of the operator L: $(L_0^{p,m,1})^M\to(L^p)^M$ given by (1):

$$(2)\qquad Lu=(I+K)((-1)^{m/2}\Delta^{m/2}+\partial/\partial t)Iu,$$

where I is the identity matrix;

$$((-1)^{m/2}\Delta^{m/2} + \partial/\partial t)Iu = ((-1)^{m/2}\Delta^{m/2}u_j + (\partial/\partial t)u_j),$$

and

$$\Delta u_j = \sum_{k=1}^{M} (\partial^2/\partial x_k^2)\, u_j,$$

m is assumed to be an even number, and finally $K = K_1 + K_2$, $K_i = (K_i^{k,j}(x, t; y, s))$ where

$$K_i^{k,j}(u) = \lim_{\epsilon\to 0} \int_0^{t-\epsilon} \int_{E^n} K_i^{k,j}(x, t; x - y, t - s)u(y, s)\, dy\, ds,$$

$$\text{for } u \in L^p(E^n x(0, R)).$$

Here K_1 is a matrix of singular integral operators, as defined in Abstract 65T-69, Notices Amer. Math. Soc. **12** (1965); while:

$$\|K_2\|_1 = \operatorname*{ess\ sup}_{(x,t)\ E^n x(0,R)} \sum_{k,j} \int_0^R \int_{E^n} \left| K_2^{k,j}(x, t; y, s)\right| dy\, ds < \infty.$$

Therefore $I+K$ is a bounded operator from $(L^p(E^n x(0, R)))^M$ into itself.

The proof consists in showing that $I+K$ is actually an isomorphism from $(L^p(E^n x(0, R)))^M$ onto $(L^p(E^n x(0, R)))^M$ and hence the problem is reduced to studying the operator $((-1)^{m/2}\Delta^{m/2}+\partial/\partial t)I$.

The theorem establishes existence and uniqueness for generalized solutions with initial condition zero. For the general initial value problem, the decomposition (2) reduces the problem to the operator $((-1)^{m/2}\Delta^{m/2}+\partial/\partial t)I$, with the same initial condition.

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