THE DIRICHLET PROBLEM FOR THE MINIMAL SURFACE EQUATION, WITH INFINITE DATA

BY HOWARD JENKINS AND JAMES SERRIN[1]

Communicated by L. Nirenberg, September 10, 1965

It is well known that there exists a unique solution of the minimal surface equation

$$(1 + q^2)r - 2pqs + (1 + p^2)t = 0$$

in a bounded convex domain D, taking arbitrarily assigned continuous data on the boundary of D. On the other hand, the celebrated solution of H. F. Scherk [6], given by the function

$$u = \log \cos x - \log \sin x,$$

takes the boundary values plus infinity on the vertical sides of the square $|x| < \pi/2$, $|y| < \pi/2$, and the boundary values minus infinity on the horizontal sides. This suggests the possibility of posing a boundary value problem for the minimal surface equation in which infinite data is assigned on certain boundary arcs of D. It is a consequence of previous results of the authors (cf. [3, Lemma 6]) that if u is a solution in a convex domain D which assumes the value plus infinity or minus infinity on a boundary arc of D, then the arc must necessarily be straight. This being the case, the most general boundary value problem with infinite data takes the following form. Let D be a bounded convex domain whose boundary contains two families of open straight segments $A_1, \cdots, A_k$ and $B_1, \cdots, B_l$, such that no two segments A_i and no two segments B_i have common endpoints. The remainder of the boundary then consists of open convex arcs $C_1, \cdots, C_m$ and endpoints of the segments A_i and B_i. *It is now required to find a solution of the minimal surface equation in D which takes the value plus infinity on each segment A_i, the value minus infinity on each segment B_i, and assigned continuous* (*though not necessarily bounded*) *values on the remaining arcs C_i.* The solution of Scherk, for example, corresponds to the case where D is a square with plus infinity assigned on the horizontal sides and minus infinity assigned on the vertical sides, the family $\{C_i\}$ being empty.

Notwithstanding this example, one might at first suppose that the problem as stated is not well posed. This turns out, however, not to

[1] This work was partially supported by the Air Force Office of Scientific Research under Grants No. AF-AFOSR-883-65 and AF-AFOSR-372-63.

be the case. We shall in fact give simple necessary and sufficient conditions for the problem to be solvable, conditions which depend only on the relative geometry of the arcs A_i and B_i.

Let $\mathcal{P}$ be a simple closed polygon whose vertices are drawn from among the endpoints of the segments A_i and B_i. Let α denote the length of $\mathcal{P} \cap \{A_i\}$, let β denote the length of $\mathcal{P} \cap \{B_i\}$, and let the perimeter of $\mathcal{P}$ be denoted by γ. Then the following results hold.

THEOREM 1. *Assume that the family $\{C_i\}$ is nonempty. Then there exists a solution of the minimal surface equation in D which assumes the value plus infinity on each A_i, minus infinity on each B_i, and assigned continuous values on each C_i, if and only if*

$$2\alpha < \gamma \quad \textit{and} \quad 2\beta < \gamma$$

for each polygon $\mathcal{P}$ of the type defined above. The solution is unique if it exists.

THEOREM 2. *If the family $\{C_i\}$ is empty, then there exists a solution of the minimal surface equation in D which takes the value plus infinity on each A_i and minus infinity on each B_i if and only if*

$$2\alpha < \gamma \quad \textit{and} \quad 2\beta < \gamma$$

for each polygon $\mathcal{P}$ which with its interior is properly contained in $\overline{D}$, and

$$\alpha = \beta$$

for the polygon $\mathcal{P}$ coinciding with the boundary of D. If it exists, the solution is unique up to an additive constant.

Some special cases are of interest. If D is a convex quadrilateral with sides A_1, C_1, A_2, C_2 in that order, then the necessary and sufficient condition for a solution to exist reduces simply to

$$|A_1| + |A_2| < |C_1| + |C_2|,$$

that is, the sum of the lengths of the sides on which infinite data is prescribed should be less than the sum of the lengths of the sides on which continuous data is prescribed. If the sides of D are A_1, B_1, A_2, B_2 in that order, then the necessary and sufficient condition for a solution to exist becomes

$$|A_1| + |A_2| = |B_1| + |B_2|.$$

We note further that a regular $2n$-gon with boundary values plus infinity and minus infinity assigned on alternate sides also satisfies the conditions of Theorem 2. The simplest case of Theorem 1, in which there is only one A_i and the data is continuous on the remainder of the boundary has been noted by R. Finn (cf. [1, Theorem VIII. 1]).

We outline here the existence proof for the special case of Theorem 1 when the family $\{B_i\}$ is empty. The demonstration rests heavily on the following result obtained in [3].

MONOTONE CONVERGENCE THEOREM. *Let $\{u_n\}$ be a monotonically increasing sequence of solutions of the minimal surface equation in a domain D. If the sequence is bounded at a single point of D, then there exists a nonempty open subset U of D such that $\{u_n\}$ converges to a solution in U, and diverges to infinity on the complement of U. Moreover, the boundary of U consists of interior chords of D and arcs of the boundary of D.*

Further information on the structure of the domains of convergence and divergence can be obtained by studying the conjugate function ψ, which arises by integrating the exact differential

$$d\psi = \frac{p}{W}dy - \frac{q}{W}dx$$

corresponding to a given solution u (here $p = u_x$, $q = u_y$, and $W = (1+p^2+q^2)^{1/2}$ in the usual notation). We first observe that $|\nabla\psi| = |\nabla u|/W < 1$ in the domain D of a solution. Thus if C is a piecewise smooth curve lying in the closure of D, we have

$$\left|\int_C d\psi\right| \leqq |C|. \tag{1}$$

It can be shown further that if C is a *convex* boundary arc of D, and if u is continuous in $D \cup C$, then

$$\left|\int_C d\psi\right| < |C|. \tag{2}$$

On the other hand, using techniques developed in [2], [3], [4], to estimate the gradient of a solution u, we have

$$\int_T d\psi = |T| \tag{3}$$

for any positively oriented boundary segment T of D where u takes on the value plus infinity (this relation is heuristically evident from the behavior of the gradient of u near T). Similarly, if $\{u_n\}$ is a sequence of solutions which are continuous in $D \cup T$, and if the sequence remains uniformly bounded on compact subsets of D while diverging uniformly to infinity on T, then

$$\lim_{n\to\infty} \int_T d\psi_n = |T|. \tag{4}$$

The same conclusion holds also if the sequence remains uniformly bounded on compact subsets of T, while diverging uniformly to infinity on compact subsets of D.

By integrating $d\psi_n$ around suitably chosen closed contours in $\overline{D}$, and using the evident condition $\oint d\psi_n = 0$, the following results supplementary to the monotone convergence theorem are easily established.

1. No component of the set of divergence can consist of a single interior chord of D.

2. Two interior chords of D which form part of the boundary of U cannot have a common endpoint.

3. Let C be an open convex boundary arc of D. Then an interior chord of D which bounds U cannot terminate at C if (i) the functions u_n are continuous in $D \cup C$ and uniformly bounded on C, or if (ii) the functions u_n are continuous in $D \cup C$ and diverge uniformly to infinity on C.

Turning to the proof of Theorem 1 in the case where the family $\{B_i\}$ is empty, let f denote the assigned continuous data on the arcs C_i. Let u_n be the solution of the minimal surface equation in D such that $u_n = n$ on each segment A_i and $u_n = \mathrm{Min}(n, f)$ on each arc C_i. The existence of the functions u_n follows directly from the results of Nitsche [5], or Finn [1]. Now by the generalized maximum principle (cf. [5]) the sequence $\{u_n\}$ is monotonically increasing in D. The monotone convergence theorem is therefore applicable, and the sequence $\{u_n\}$ converges to a solution u in a (possibly empty) open set U in D. Let V denote the complementary set. The results above imply that any component of V must be bounded by a simple closed polygon $\mathcal{P}$ whose vertices are among the endpoints of the segments A_i. Letting $\mathcal{A} = \mathcal{P} \cap \{A_i\}$ it follows from (4) that

$$\lim_{n\to\infty} \left| \int_{\mathcal{P}-\mathcal{A}} d\psi_n \right| = \gamma - \alpha.$$

On the other hand, from (1),

$$\lim_{n\to\infty} \left| \int_{\mathcal{A}} d\psi_n \right| \leqq \alpha.$$

But $\oint_{\mathcal{P}} d\psi_n = 0$, and this contradicts the assumption $2\alpha < \gamma$. Thus the set V must actually be empty, and $\{u_n\}$ converges to a solution in all of D. That the limit function assumes the value plus infinity on each segment A_i is obvious, and that it takes the values f on each C_i follows from a barrier-type argument.

The condition $2\alpha < \gamma$ can easily be shown necessary for the existence of a solution, using the relations (2) and (3).

To obtain the general case of Theorem 1, and also to prove Theorem 2, we again use approximating sequences of solutions $\{u_n\}$. In these cases, however, it is not convenient to construct the sequences so that they are monotonically increasing. Rather we use the special case whose proof is outlined above to establish the existence of solutions which majorize the approximating sequences. The required conclusions are then obtained by compactness and barrier arguments.

Added in Proof. If the boundary value problem is to be solvable for arbitrary continuous data, then the domain must be convex. (cf. the famous example of H. A. Schwarz.) Nevertheless, Theorems 1 and 2 can be generalized essentially without change to nonconvex and even multiply connected domains which are bounded by families of convex arcs. Thus once one admits the possibility of discontinuities in the data at a finite set of boundary points, it is no longer necessary to retain the hypothesis of convexity in order to have a well-posed problem. The complete proofs of Theorems 1 and 2, and the details of the generalization to nonconvex domains will appear elsewhere.

References

1. R. Finn, *New estimates for equations of minimal surface type*, Arch. Rational Mech. Anal. 14 (1963), 337–375.

2. E. Hopf, *On an inequality for minimal surfaces $z = \phi(x, y)$*, J. Rational Mech. Anal. 2 (1953), 519–522.

3. H. Jenkins and J. Serrin, *Variational problems of minimal surface type*. I, Arch. Rational Mech. Anal. 12 (1963), 185–212.

4. J. C. C. Nitsche, *Über eine mit der Minimalflächengleichung zusammenhängende analytische Funktion und den Bernsteinschen Satz*, Arch. Math. 7 (1957), 417–419.

5. ———, *Über eine verallgemeinertes Dirichletsche problem für die Minimalflächengleichungen und hebbare unstetigkeiten ihrer Lösungen*, Math. Ann. 158 (1965), 203–214.

6. H. F. Scherk, *Bemerkungen über die kleinste Fläche innerhalb gegebener Grenzen*, J. Reine Angew. Math. 13 (1835), 185–208.

University of Minnesota