BOOK REVIEWS

Modern algebraic topology. By D. G. Bourgin. Macmillan, New York, 1963. xiii+544 pp. $11.50.

The volume under review must surely constitute the most comprehensive study of algebraic topology in the literature. Admittedly this is a very big book, having 544 pages; but, even so, it is a remarkable feat to have assembled in that span, starting from scratch, a discussion of the techniques and methodology, and a statement of the principal results in all the basic areas of homology theory, except for obstruction theory, cohomology operations and extraordinary cohomology.

The book is divided into 17 chapters and an appendix devoted to point-set topology. Chapter 1 (*Preliminary algebraic background*) gives the most elementary relevant algebraic definitions. In Chapter 2 (*Chain relationships*) the chain groups of a simplicial complex are defined, and in Chapter 3 (*Fundamentals of the absolute homology groups and basic examples*) the absolute homology groups of finite complexes are introduced, some computations made and pseudomanifolds defined.

Chapter 4 (*Relative omology modules*) consists in the main of an injection of a further quantity of basic algebra—vector spaces, modules, direct sums and products, graded modules and algebras, chain modules, exact sequences, cochains, cohomology. There is also a discussion of the dual complex. The term "omology" to describe the concept of which homology and cohomology are manifestations makes its remarkable appearance in this chapter. Simplicial manifolds are defined in Chapter 5 (*Manifolds and fixed cells*) and Poincaré duality is proved for them. A geometrical interpretation is given of cocycles and the chapter closes with a treatment of the Lefshetz number of a self-chain-map.

Chapter 6 (*Omology exact sequences*) provides yet another infusion of algebra. The exact homology sequence is obtained from a short exact sequence of chain complexes and applied to obtain the Mayer-Vietoris sequence and the exact sequence of a triple. There are also sections devoted to chain homotopy and to tensor products (over an integral domain).

Chapter 7 (*Simplicial methods and inverse and direct limits*) takes up the question of the invariance of the homology groups under subdivision and hence defines, through the simplicial approximation theorem, the homology homomorphism induced by a continuous map. The final three sections of the chapter contain a definition of a normal (or regular) neighborhood of one complex in another, an extended treatment of inverse and direct systems with special reference to inverse systems of compact groups, and a discussion of fixed point theorems and coincidences. In Chapter 8 (*Gratings*) the notion of a grating on the space X (a ring-complex with elements supported by closed subsets of X) is introduced as a means of extending the cohomology notions to arbitrary topological spaces. Special attention is given to the singular (simplicial and cubical) gratings, the Alexander grating, and the Čech grating. This discussion leads in Chapter 9 (*Fundamental omology relations and applications*) to a treatment of the continuity axiom, here appearing as a theorem for Alexander and Čech cohomology, and of excision. There is a discussion of manifolds, followed by some geometrical applications and the introduction of the degree of a map.

Chapter 10 (*Homological algebra*) deals with more sophisticated algebraic notions than have so far appeared. Indeed most of the ideas which would be regarded as fundamental to an introductory course in homological algebra, together with some more specialized ones, appear here—projective and injective modules, resolutions, Tor and Ext, categories and functors, derived functors, adjoint functors, Künneth theorems and universal coefficient theorems. Tucked away at the end are the Eilenberg-Steenrod axioms for omology.

Chapter 11 (*Uniqueness proofs and fixed point indices*) opens with a uniqueness theorem expressed in terms of compact gratings on locally compact spaces. This is followed by a definition and discussion of the fixed point index for smooth spaces (that is, spaces covered by a family of compact sets whose finite intersections are all acyclic). Chapter 12 (*Products*) deals with cup products, cross products, and cap products in the omology theories under study. Steenrod's $\cup_i$-products also appear. There is then some discussion of topological coefficient groups and the Pontryagin duality theorem in the compact-discrete case, followed by treatments of Alexander duality, the linking coefficient, and Hopf manifolds. These last are generalizations of Hopf's Gruppenmannigfaltigkeiten and the Hopf theorem on their cohomology with rational coefficients is proved. There is then a substantial section devoted to Hopf algebras, along the lines of Milnor-Moore.

In Chapter 13 (*Groups of homeomorphisms*) the theory of transformation groups is developed. Equivariant homology groups are defined and there is a special discussion of sphere-maps, leading to the Borsuk-Ulam Theorem and its generalization by Bourgin-Yang. There is also a pretty full treatment of the Smith theory. The theory of fibre spaces is taken up in Chapter 14 (*Fiberings*); fibre bundles, principal fibre bundles (though the universal bundles do not appear till the end of Chapter 17), and Serre fibre spaces are presented, with the theoretical discussion centering on the lifting homotopy property. The theory of the fundamental group and covering spaces also appears here, as well as a further discussion of fixed points.

Chapter 15 (*Homotopy*) opens with the definition of the higher homotopy groups and the homotopy sequences for pairs and triads are established. There is a section on CW-complexes and some discussion of the suspension homomorphism, of path-spaces and loop-spaces, and of the Cartan-Serre-Whitehead procedure for killing homotopy groups. A final section introduces the Barratt-Eckmann-Hilton "extended" homotopy groups and mentions the category for Spanier-Whitehead S-theory.

Chapter 16 (*Spectral sequences*) is devoted almost exclusively to the development of the algebraic foundations of the theory of spectral sequences. Applications all appear at the end of Chapter 17 (*Sheaf theory*) where, after a rather full account of cohomology theory with coefficients in a sheaf, the Leray spectral sequence is described, and (by confining attention to constant sheafs) an elementary application of spectral sequences to Serre fibrations is given.

This somewhat sketchy synopsis sufficiently indicates the very impressive scope of the book. It is sure that any reader who has mastered its contents will be able to launch himself into more specialized study in algebraic topology and even to embark on research in the field. However, this reviewer is left in great doubt as to whether any reader will master the contents or whether, having done so, he will know what to look for in his further reading and research.

We take up the second point first. It does not seem that the motivation is always clear for the topics under discussion. A notable exception to this is the topic of fixed points and coincidences, which threads its way through the book and provides purpose for more fundamental concepts. But, especially in the later algebraic chapters, on homological algebra and spectral sequences, it must seem to the reader that a tremendous battery of technique is being arrayed to rather little purpose. Chapter 16 is virtually all technique and definition and the part of Chapter 14 devoted to general fibre spaces is mainly definition; yet the pay-off for the effort the reader must have put in to master all these ideas is extremely meagre. Perhaps even more fundamentally one might claim that the significance of homology as a homotopy-invariant functor to graded abelian groups is never given sufficient stress, and that no real reason is offered against the student deciding it would be preferable to go into another branch of mathematics requiring a less arduous apprenticeship.

A second practical difficulty in connection with further reading is that there are no references to papers and other sources in the body of the text, nor does the list of articles indicate the connections with specific parts of the book. The list is, moreover, sadly deficient. It is very surprising, for example, to find no mention of Serre's thesis on fibre spaces nor of Massey's article introducing exact couples. The author uses the Cartan-Leray approach, rather than the Godement approach, to sheaves, but omits a reference to Cartan's or Gray's exposition. He is generous enough to include a special section in Chapter 15 on the Barratt-Eckmann-Hilton "extended" homotopy groups but has no reference to the relevant work of these authors. CW-complexes also occupy a section but, perhaps most astonishing of all, J. H. C. Whitehead does not rate a mention.[1]

We turn now to a consideration of the exposition itself. Before proceeding, however, we wish to make it clear that we appreciate that many of the critical remarks we will make may well apply with equal force to other books dealing with the same topic and that, quite emphatically, no comparisons are intended.

The book then, let it be said, made very hard and often painful reading. It is difficult to find one's way around in it. The sections of the chapters are not listed in the table of contents and the index is highly lacunary.[2] Moreover the reviewer even had difficulty in saying just when certain theorems had been proved. This was true of the fundamental theorem of Serre on the homology of fibre spaces and of the theorem that simplicial homology theory is a homology theory in the axiomatic sense. Let us consider more explicitly the question of the homotopy invariance of the homology groups. Homotopy type is defined in Chapter 13 (p. 316), but by then the reader has "understood" that homology is homotopy invariant. The notion of homotopy itself is defined on p. 178, although it appears in the formulation of Theorem 5.5 on p. 120.

The definitions would often be found by the student extremely difficult to master, for a variety of reasons. In some cases (e.g., Definition 1.2, p. 72; Definition 8.1, p. 414) there was no indication what was being defined! In other cases there were syntactical obscurities; thus in defining a sheaf (A, p, X) on p. 463 we find (with $A(x) = p^{-1}x$)

[1] There is a disclaimer at the start of the bibliography that draws some of the sting of this general criticism of omissions. The reviewer however cannot accept that it is justifiable to suppress all mention of these, and other, basic and highly accessible papers.

[2] Though very usefully supplemented by an index of symbols.

"(1.1c) The operations on A, consequent on the structure of $A(x)$ are required to be continuous. Thus the homomorphism $a \rightarrow ra$ on $A(pa)$ to $A(pa)$ is continuous as a varies over A. Also algebraic addition is continuous. We spell this out. Suppose $pa = pb$ and suppose $W(a-b)$ is an assigned neighborhood in A of $a-b$. Then there exist neighborhoods $U(a)$ and $V(b)$ such that $pa' = pb' \in pU(a) \cap pV(b)$ and if $a' \in U(a)$ and $b' \in V(b)$, then $a'-b' \in W(a-b)$."

Some definitions are unnecessarily special.[8] Modules and tensor products are only defined over integral domains; no gain in simplicity results. Moreover in exhibiting a projective module which is not free this restriction is abandoned. Even more curious, the one place where the restriction is used is in providing a proof that every module over a principal ideal domain can be embedded in an injective module, a proof which does not work in the general case—and which is slightly more involved than the elegant elementary argument of Eckmann-Schopf. In one or two cases the definitions contain errors which do not seem to be attributable to misprints. Thus (Definition 7.1, p. 363) $p: X \rightarrow B$ is a covering map of the connected, locally-connected space B if it is onto and each point of B has a neighbourhood, the components of whose counterimage are mapped homeomorphically. The author adds, "The next lemma might have been made a part of the definition. Lemma 7.2. In a covering space, B also is connected and locally-connected." The presence of "also" and the actual definition make it appear probable that "B" should be replaced by "X"; but then the lemma does not follow from the definition.

Few of the difficulties encountered were due to actual identifiable errors; in many more cases it was a matter of obscurity arising from syntactical idiosyncrasies, infelicities of style, the mixing or confusing of notations, or mere misprints. (The presence of more than one of these factors in the same formulation often proved particularly vexing.)

Among the errors which will disturb the student for whom the text is intended, if not the sophisticated reader, we mention the following. On p. 7 it is stated, "This section concerns itself exclusively with integer matrices $\cdots$. A square matrix is nonsingular if $\Delta(A) \neq 0$. The nonsingular square matrix A has an inverse denoted by $A^{-1} \cdots$." On p. 194 a Dedekind ring is defined in such a way as to imply that

[8] Other definitions are unexpectedly general. Thus (Def. 7.3, p. 221) a cohomology functor from an exact category to an additive category is a sequence $\{T^n\}$ of covariant (*sic*) functors converting a short exact sequence of chain-complexes into a long sequence of order 2.

it is a unitary ring in which every ideal contains the unity. On p. 205 the universal coefficient theorem $H_p(K, G) \approx H_p(K) \otimes G + \text{Tor}\,(H_{p-1}(K), G)$ is given, followed by the sentence "If G is a field the Tor term drops out of course."

It would ordinarily be perverse to devote any substantial proportion of a review of a mathematical text to a discussion of style. However, in this case the author is evidently at pains to expose with clarity and precision what he and the reviewer regard as an important part of mathematics. The effort he has put into this book is prodigious and it is with a view to pointing to improvements which the reviewer hopes to see made in subsequent editions that we now discuss and document certain obscurities to which reference has already been made.

Already on p. 1, where groups are defined, we have R^1 denoting "the real numbers excepting 0." Lower down the paragraph, in talking of the operation of addition, the author has " . . . and now R^1 is understood to include 0 which plays the role of 1." The student who *needs* to be told what a group is is apt to be mystified by such a formulation. On p. 12 we find "The incidence number denoted by $[\sigma_q^a, \sigma_r^b]$ or $\eta_b^a(q)$ takes on the values 0, ± 1 and is non 0 only if $\sigma_{r=q-1}^b < \sigma_q^a$"—a propositional afterthought appearing in the suffix! Definition 4.1 on p. 102 has F as a well-defined free Abelian group, followed by an unuseful formula for F (marred by a misprint), followed by the phrase "Let M be the smallest Abelian group generated by . . . "; then on p. 103 F suffers a bihomomorphism; we found no definition of this, nor is it mentioned in the index, nor do we believe it afflicts Abelian groups. Studying p. 192, the reader will surely be worried by

> "The extent of the generalization achieved is given by the following lemma.
>
> LEMMA 2.4. *A module, P, is projective iff P is a direct summand of a free module, F.*
>
> LEMMA 2.5. *An injective module is a direct summand of any containing module.*
>
> Sufficiency follows immediately. . . . "

It is *Lemma* 2.4 *alone* which generalizes something and whose proof begins in the quotation above.[4] On p. 207, after Ext (C, G) is defined through a presentation of C and before the diagrammatic representation of this definition by means of an exact sequence, there occurs the solitary sentence, surely baffling to the beginner, "Accordingly

[4] Although the proof of Lemma 2.5 follows immediately, initiated by the phrase "The last assertion · · · ."

Ext (C, G) is the group of extensions of C by G." (In the index, "extension" refers only to extension of maps.) Axiom IV on p. 214 seems to identify all zero maps in a category; and on p. 217 in two adjoining and closely related paragraphs A appears as the direct sum of a collection A_λ and as a container of subobjects A_i, whose sup is going to be constructed through the intermediary of their direct sum. The fact that A is used in these two different senses is masked by the fact that it appears once in the second paragraph as $\mathfrak{A}$—the symbol for the category to which it belongs. As a final and particularly puzzling example we refer to the proof of Theorem 8.8 on p. 417 which contains the passage

"As a trick for exposition we denote somewhat inaccurately by $p\otimes\phi$ and $\phi\otimes p$ the obvious projections on $Y\times Y$ to Y. Effectively ϕ acts as a 'space annihilator,' $\otimes$ as an indexing device and p is used for 1."

The reviewer is on particularly vulnerable ground in criticizing notation. However the criteria of inner consistency, absence of gratuitous conflict with existing notation, and helpfulness in elucidation are surely not arguable. This review is already too full of examples, but it is difficult in any other way to point to what are, in the reviewer's opinion, violations of these criteria. A simple example occurs on p. 213. Definition 6.2 reads "The morphism $\alpha\in G(A, B)$ is an equivalence if there is a map $\alpha': B\rightarrow A$ such that . . . ." Categories have just been defined and are, presumably, unfamiliar; the notation "$\alpha': B\rightarrow A$" has not been introduced. What will the reader think of this juxtaposition of "the morphism $\alpha\in G(A, B)$" with "a map $\alpha': B\rightarrow A$"? Will he suspect them of being extremely similar things? On p. 361 the notation $\Omega(B)$ denotes what we call B^I, the loop-space on B (at b_0) being Λ_{b_0}; this latter notation is in marked contrast to that of ΣX for the suspension on p. 409 (where, incidentally, ΣX is defined to be the unreduced and reduced suspension within the same paragraph without mention of the distinction). Perhaps the most striking example of confusing notation (side-by-side with the downright opaque) occurs in the definition of the Whitehead product on pp. 387–8. On p. 385 (fg) is defined and its homotopy class denoted by $\boldsymbol{fg}$. On p. 388 a new (fg) is defined (with no mention of the old) whose homotopy class is written $\boldsymbol{f}\circ\boldsymbol{g}$. The problem immediately following asks the reader to show

$$\boldsymbol{fg} = (\boldsymbol{g}\circ\boldsymbol{f})^{(-1)^{mn}} = \boldsymbol{fg}^{-1} \quad \text{for } m = 1, n > 1.$$

(Even the author here seems to be sunk by his own notation!) The definition of the Whitehead product (Definition 1.6) indeed has baffled many who thought themselves familiar with homotopy theory. Among its prescriptions, it asks us "Write, following (1.2a), $I^{m+n} = I^m \times I_m^{m+n}, \cdots$." (1.2a) turns out to be (with the misprint corrected)

$$X^{I^n}, A^{\dot{I}^n}, x_0^{J^{n-1}} = X^{I^k \times I_k^n}, A^{\dot{I}^k \times I_k^n \cup I^k \times \dot{I}_k^n}, x_0^{J^{n-1}}.$$

It was some time before the reviewer realized that "following (1.2a)" refers to what *follows* (1.2a) and not (1.2a) itself!

Misprints are excessively numerous. Of particular importance are the following. There are two on p. 7. First the explanation of the symbol a_i^j is flatly contradicted by the instance given; second we have the unenlightening statement that the 0 matrix $\mathbf{0}$ is the square matrix $\mathbf{0} = (0)$. Then on p. 201 the meaning of Definition 4.2 is obscured by having $\langle \boldsymbol{h} \otimes \boldsymbol{g} \rangle$ instead of $\langle h \otimes g \rangle$. On p. 215 a delicate point is lost because in a discussion involving ψ and Ψ, many ψ's appear as Ψ. On p. 412 we can only assume that Lemma 6.4 is a misprint, although we have not reconstructed it (perhaps it would assert that A is a deformation retract of $\Omega(A, Z)$); and on pp. 415, 416, amid other misprints, there occur some highly confusing replacements of X, Y by x, y. The bibliography misspells the names Eckmann, Hirzebruch and Wylie; and the reference to Hopf's paper on group-manifolds has evidently been through an unusually efficient scrambler.

To sum up, the reviewer admires the sweep and coverage achieved by the author; he and the author would have chosen differently from the supply of special topics to illumine the basic material, but that is surely no criticism. The reviewer would have preferred less "general topology" to make room for the cohomology topics listed at the start of this review, but this is just a matter of taste. The reviewer's real disquiet springs from his feeling that the text before him is not yet thoroughly ready for publication and requires substantial emendation and editing along the lines indicated. He trusts that his criticisms may be interpreted in this constructive light and that a new and greatly improved edition of this book may appear.

PETER HILTON

The foundations of intuitionistic mathematics. By Stephen Cole Kleene and Richard Eugene Vesley.

This book consists of four chapters, three by Kleene and one by Vesley. The authors' general purpose is to formalize a portion of intuitionistic analysis and to pursue certain investigations within and certain investigations about the formal system. Such an enterprise, however admirable the mathematics involved, may not be sym-