ASYMPTOTIC VALUES OF HOLOMORPHIC FUNCTIONS OF IRREGULAR GROWTH

BY ALFRED GRAY AND S. M. SHAH[1]

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Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be holomorphic with radius of convergence R $(0<R\leqq\infty)$, and let $\mu(r)$ denote the maximum term and $\nu(r)$ the central index of $f(z)$. By definition, for $r>0$, $\mu(r)=\max\{|a_n|r^n|n=0,1,2,\cdots\}$ and so $\mu(r)=|a_{\nu(r)}|r^{\nu(r)}$. We shall assume that $\mu(r)\to\infty$ as $r\to R$ and $f(z)$ is not a polynomial. In this note we give a technique for comparing $f(z)$ with its maximum term which shows that, for certain functions $f(z)$ which are of very slow growth, or whose power series have wide gaps, $f(z)$ has no finite asymptotic values. Our result is to be compared with Wiman's theorem [1, Chapter 3], [5]: If $f(z)$ is an entire function of order $\rho<\frac{1}{2}$ then $f(z)$ has no finite asymptotic values. However, the class of functions for which we show the nonexistence of finite asymptotic values is different from that of Wiman; in particular we allow the functions to have a finite radius of convergence.

Let $z=re^{i\theta}$ and define

$$\mu(re^{i\theta}) = \mu(r)e^{i\nu(r)\theta}$$

for $r>0$ and $0\leqq\theta<2\pi$. Then $\mu(z)$ is a complex extension of $\mu(r)$; it is piecewise continuous, but has discontinuities where $\nu(|z|)$ is discontinuous.

Let $\gamma(t)$ be a (continuous) receding curve such that $|\gamma(t)|\to R$ as $t\to\infty$. Then $\gamma(t)$ is an *asymptotic path* of $f(z)$ if as $t\to\infty$, $f(\gamma(t))$ tends to a limit ω, called an *asymptotic value*; analogously with this definition we shall call $\gamma(t)$ a *μ-asymptotic* path if $f(\gamma(t))/\mu(\gamma(t))$ tends to a limit ω as $t\to\infty$, and we say that ω is a *μ-asymptotic value.* For example, e^z has μ-asymptotic value ∞ along the positive real axis, but has μ-asymptotic value 0 along any path to ∞ in any angle which excludes the positive real axis. The following theorem is obvious, since $\mu(r)\to\infty$.

THEOREM 1. *If $\gamma(t)$ is an asymptotic path of $f(z)$ with finite asymptotic value, then $\gamma(t)$ is a μ-asymptotic path of $f(z)$ with μ-asymptotic value* 0.

Next we investigate some situations in which $f(z)$ has no μ-asymp-

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totic values. Without loss of generality assume $a_0 \neq 0$. Let $\{\rho(n)\}$ be the sequence of jump points of $\nu(r)$, counting multiplicity. Since $\nu(r) \to \infty$ as $r \to R$, $\rho(n) \to R$ as $n \to \infty$. We denote by $\{n_k\}$ the range of $\nu(r)$, so that $\nu(\rho(n_k)) = n_k$, and we define $n_0 = 0$. Then $0 < \rho(n_k) < \rho(n_k+1) = \cdots = \rho(n_{k+1}) < \cdots$.

Explicitly

$$\rho(n_k) = \left| \frac{a_{n_{k-1}}}{a_{n_k}} \right|^{1/(n_k - n_{k-1})}.$$

We define

$$L = \limsup_{k \to \infty} \frac{\rho(n_{k+1})}{\rho(n_k)},$$

$$S = \limsup_{k \to \infty} (n_{k+1} - n_k),$$

$$\left.\begin{matrix}\Phi \\ \phi\end{matrix}\right\} = \lim_{k \to \infty} \begin{matrix}\sup \\ \inf\end{matrix} \, (n_{k+1} - n_k) \log \left\{ \frac{\rho(n_{k+1})}{\rho(n_k)} \right\},$$

$$\left.\begin{matrix}\Xi \\ \xi\end{matrix}\right\} = \lim_{k \to \infty} \begin{matrix}\sup \\ \inf\end{matrix} \, (n_k - n_{k-1}) \log \left\{ \frac{\rho(n_{k+1})}{\rho(n_k)} \right\}.$$

The proofs of the following theorems are given in [2], [3], and [4].

THEOREM 2. *If $L > 1$ and $S < \infty$, then $f(z)$ has no μ-asymptotic values. (The hypothesis $L > 1$ implies that $f(z)$ is a transcendental entire function.)*

THEOREM 3. *Suppose that $f(z)$ has the form $f(z) = \sum_{k=0}^{\infty} a_{n_k} z^{n_k}$ where $\{n_k\}$ is the range of $\nu(r)$. If any of the conditions* (1)–(4) *hold, then $f(z)$ has no μ-asymptotic values.*

$$\phi = \Xi = \infty, \quad \xi > 0. \tag{1}$$

$$\xi = \Phi = \infty, \quad \phi > 0. \tag{2}$$

$$R = \infty, \quad \sum_{k=1}^{\infty} \frac{1}{n_{k+1} - n_k} < \infty, \quad \phi = \infty, \quad \xi > 0. \tag{3}$$

$$R = \infty, \quad \sum_{k=1}^{\infty} \frac{1}{n_{k+1} - n_k} < \infty, \quad \xi = \infty, \quad \phi > 0. \tag{4}$$

THEOREM 4. *Suppose that $f(z)$ has the form*

$$f(z) = \epsilon(0) + \sum_{k=1}^{\infty} \frac{\epsilon(k) z^{n_k}}{\rho(1) \cdots \rho(n_k)}.$$

Assume $|\epsilon(k)| = 1$ and $\epsilon(k)$ has period h where h is a positive integer. If $0 < \phi = \Phi < \infty$ and $0 < \xi = \Xi < \infty$, then $f(z)$ has no μ-asymptotic values.

Theorem 1, combined with Theorems 2, 3 and 4, yields

THEOREM 5. *If the hypotheses of Theorems* 2, 3, *or* 4 *are satisfied, then* $f(z)$ *has no finite asymptotic values.*

EXAMPLES. Each of the following functions has no finite asymptotic values:

$$\sum_{k=0}^{\infty} \frac{z^k \exp i\alpha_k}{\lambda^{(1/2)k(k+1)}}, \qquad \sum_{k=0}^{\infty} \frac{z^{p^k-1} \exp i\alpha_k}{\Gamma(\alpha p^k + 1)},$$

$$\sum_{k=0}^{\infty} \frac{z^{p^k} \exp i\alpha_k}{\{(k+\lambda)\log p\}^{p^k}}, \qquad 1 + \sum_{k=1}^{\infty} \frac{z^{k^p}}{\Gamma(\alpha k^p + 1)},$$

$$1 + \sum_{k=1}^{\infty} \frac{z^{k^q}}{\{(k+\lambda)\log q\}^{k^q}}, \quad \text{and} \quad \sum_{k=1}^{\infty} \exp\left(\frac{p^{k\beta}}{\beta}\right) z^{p^k-1},$$

where $\lambda > 1$, $\alpha > 0$, $0 < \beta < 1$, $0 \leqq \alpha_k < 2\pi$, p is an integer greater than 1, and q is an integer greater than 2.

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UNIVERSITY OF CALIFORNIA, BERKELEY AND
UNIVERSITY OF KANSAS