KERNEL FUNCTIONS AND NUCLEAR SPACES

BY JOSEPH WLOKA

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As is well known, it is possible to represent any complete, locally convex space E as

$$(1) \qquad E = \underset{\leftarrow \alpha \in A}{\text{proj}} E_\alpha,$$

where the E_α are Banach spaces. If the projective mappings of (1) are nuclear [2], E is called a nuclear space. For complete spaces, this definition is equivalent to Grothendieck's original one (see [2], [8]). It is possible to treat nuclearity for countable inductive limit spaces in a dual fashion:

DEFINITION 1. E is called an (LN)-space if

(1) $E = \text{ind}_{\to n} E_n$, $n = 1, 2, \cdots$, where the E_n are Banach spaces,

(2) the inductive mappings (imbeddings) are nuclear.

We have the following theorem.

THEOREM 1. *Every* (LN)*-space is nuclear* (*in the sense of Grothendieck*).

For regular inductive limit spaces the inverse theorem is also true (a space $E = \text{ind}_{\to n} E_n$ is called regular, if every bounded set $A \subset E$ is already bounded in some E_{n_0}).

THEOREM 2. *If the regular space* $E = \text{ind}_{\to n} E_n$, *the* E_n *being Banach spaces, is nuclear, then it is an* (LN)*-space.*

In the above definitions and theorems, we can without loss of generality substitute Hilbert spaces H_n for the Banach spaces E_n and Hilbert-Schmidt mappings for nuclear mappings.

In what follows we need the concept of a reproducing kernel. We quote the definition of Aronszajn [1], [6]. Let H be a Hilbert space with scalar product $(\, , \,)_x$, consisting of functions $f(x)$ defined on some point set G. The function $K(x, y)$, $x \in G$, $y \in G$ is called a reproducing kernel if:

(1) for every fixed y the function $K(x, y)$ of x belongs to H,

(2) $K(x, y)$ has the reproducing property

$$f(y) = (f(x), K(x, y))_x \quad \text{for all } f \in H.$$

THEOREM 3. *Let* H_n, $n = 1, 2, \cdots$, *be a sequence of Hilbert spaces with reproducing kernels* $K_n(x, y)$, *where the scalar product in* H_n *is* *given by*

$$(\phi, \psi)_n = \int \phi(x)(\overline{\psi(x)})^- d\sigma_n(x),$$

the σ_n *being certain measures having their supports in some fixed set G and fulfilling the further condition*

$$(\phi, \phi)_n \geqq (\phi, \phi)_{n+1} \quad \text{for all } \phi \in H_n \quad (\text{i.e. } H_n \overrightarrow{\subset} H_{n+1}).$$

If, for every m, there exists an n > m such that the condition

$$\text{(K)} \qquad \int K_m(x, x)\, d\sigma_n(x) < \infty$$

holds, then the space

$$E = \underset{\to n}{\text{ind}}\, H_n,$$

is an (LN)*-space.*

REMARK. Because of Theorem 1, E is then a nuclear space. The corresponding theorem for projective limit (1), is also true: in that case the condition (K) takes the form

$$\text{(K}'\text{)} \qquad \int K_{\beta(\alpha)}(x, x)\, d\sigma_\alpha(x) < \infty, \quad \text{where} \quad \alpha, \beta \in A \quad \text{and} \quad \beta(\alpha) > \alpha.$$

In preparation for using the above theorems to establish nuclearity for spaces of holomorphic functions, let us consider the Hilbert space $H_g = \{\phi \mid \phi(z) \text{ holomorphic in } G,\ \|\phi\|^2 = \iint_G |\phi(z)|^2 |g(z)|^2 dz < \infty\}$. Here G is some open set of the complex plane and $g(z)$ is some continuous (weight) function on G different from zero. It can be shown that H_g possesses a reproducing kernel $K_g(z, w)$, continuous on G (this is a corollary of Hartogs' theorem), and satisfying the inequality

$$\text{(I)} \qquad K_g(w, w) \leqq \frac{1}{(\pi r^2)^2} \iint_{C(w,r)} |g(z)|^{-2}\, dz$$

for every disc $C(w, r)$ contained by G.

Now let G_n be a sequence of open sets (bounded or not) in the complex plane such that

$$G_n \supset G_{n+1}, \qquad n = 1, 2, \cdots,$$

and let g_n, g_n' be continuous (weight) functions $\neq 0$, defined on G_1. Let us take the Hilbert spaces $H_n = \{\phi \mid \phi(z) \text{ holomorphic on } G_n,\ \|\phi\|_{H_n}^2 = \iint_{G_n} |\phi|^2 |g_n|^2 dz < \infty\}$, and the Banach spaces

$$M_n = \left\{\phi \mid \phi(z) \text{ holomorphic on } G_n, \|\phi\|^2_{m_n} = \sup_{z \in G_n} |\phi(z) g_n'(z)| < \infty\right\}.$$

We assume that the functions g_n, g_n' are such that

$$H_n \overrightarrow{\subset} H_{n+1}, \qquad n = 1, 2, \cdots,$$

$$M_n \overrightarrow{\subset} M_{n+1}, \qquad n = 1, 2, \cdots.$$

Beside this we require g_n, g_n' to have the following properties:

For any n there exists an $m(n)$ such that

$$(\mathrm{N}_1) \qquad \begin{aligned} &1.\ m(n) \to \infty \quad \text{if} \quad n \to \infty,\ m < \infty. \\ &2.\ \iint_{G_m} \left|\frac{g_n}{g_m'}\right|^2 dz = A < \infty. \end{aligned}$$

For any $t \in G_n$ it is possible to find $d_t > 0$ such that

$$(\mathrm{N}_2) \qquad \begin{aligned} &1.\ C(t, d_t) \subset G_m, \\ &2.\ \frac{|g_n'(t)|}{\pi d_t^2}\left[\iint_{C(t,d_t)} |g_m(z)|^{-2} dz\right]^{1/2} \leqq B < \infty, \end{aligned}$$

hold for all $t \in G_n$ (n and m as in (N_1)). We can now state

THEOREM 4. *If the conditions* (N_1) *and* (N_2) *are fulfilled, the equivalence*

$$E = \operatorname{ind}_{\to n} H_n \cong \operatorname{ind}_{\to n} M_n$$

holds, and E is an (LN)-*space.*

REMARK. In proving the nuclearity we use essentially inequality (I).

If for every n, $g_n = g_n'$, these functions being holomorphic, and if the distances $d(G_{n+1}, CG_n)$ are all positive (when $CG_n = \varnothing$, $d(G_{n+1}, CG_n)$ is positive by convention) Theorem 4 follows from the single condition (N_1).

The corresponding theorem for projective limits (1) also holds under the following assumptions:

$$G_\alpha \subset G_\beta,\ H_\beta \overrightarrow{\subset} H_\alpha,\ M_\beta \overrightarrow{\subset} M_\alpha, \quad \text{for } \alpha < \beta$$

(the definitions of H_α, M_α are analogous to those of H_n and M_n)

$$g_\alpha, g_\alpha' \text{ continuous and } \neq 0 \text{ on } \bigcup_\alpha G_\alpha,$$

$$(\mathrm{N}_1') \qquad \iint_{G_\beta}\left|\frac{g_\alpha}{g_\beta'}\right|^2 dz = A < \infty \quad \text{for all } \alpha \in A \text{ and some } \beta(\alpha) > \alpha,$$

$$(\mathrm{N}_2') \qquad \begin{array}{l} 1.\ C(t, d_t) \subset G_\beta, \\ 2.\ \dfrac{|g_\alpha'(t)|}{\pi d_t^2}\left[\iint_{C(t,d_t)} |g_\beta|^{-2} dz\right]^{1/2} \leqq B < \infty, \end{array}$$

for some $d_t > 0$ and all $t \in G_\alpha$.

The above theorems can be used to obtain information about the structure of Gelfand's W- and $\mathfrak{E}$-distribution spaces, and to prove their nuclearity (for the definitions of these spaces, see [7] and [11]). These theorems also imply that Silva's ultradistribution spaces [9], and the boundary distribution spaces of Köthe [5] and Tillmann [10] are nuclear. As still another application, one can obtain simple proofs for the nuclearity of the spaces $\mathfrak{G}(G)$ considered by Grothendieck [3] and Köthe [4].

Proofs of these and other results will appear in [12].

Bibliography

1. N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1950), 337–404.

2. A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. No. 16 (1955).

3. ———, *Sur certains espaces de fonctions holomorphes*. I, II, J. Reine Angew. Math. **192** (1953), 35–64, 77–95.

4. G. Köthe, *Dualität in der Funktionen-theorie*, J. Reine Angew. Math. **191** (1953), 30–49.

5. ———, *Die Randverteilungen analytischer Funktionen*, Math. Z. **57** (1952), 13–33.

6. H. Meschkowski, *Hilbertsche Räume mit Kernfunktion*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1962.

7. W. P. Palamodow, *The Fourier transformation of quickly increasing infinitely differentiable functions*, Trudy Moskov. Mat. Obšč. **11** (1962), 309–350. (Russian)

8. D. A. Raĭkow, *On a certain property of nuclear spaces*, Uspehi Mat. Nauk **12** (1957), 231–236. (Russian)

9. J. Sebastião e Silva, *Les fonctions analytiques comme ultra-distributions dans le calcul opérationnel*, Math. Ann. **136** (1958), 58–96.

10. H. G. Tillmann, *Randverteilungen analytischer Funktionen und Distributionen*, Math. Z. **59** (1953), 61–83.

11. J. Wloka, *Über die Hurewicz-Hörmanderschen Distributionsräume*, Math. Ann. (to appear).

12. ———, *Reproduzierende Kerne und nukleare Räume*, Math. Ann. (to appear).

University of California, Los Angeles and
University of Heidelberg, Germany