THE GENUS OF K_n, $n=12(2^m)$

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Introduction. It is the object of this note to display a minimal imbedding of K_n, the complete graph on n vertices with $n=12(2^m)$, $m=0, 1, 2, \cdots$. (See [5] for notation and terminology.) A minimal imbedding of K_n with $n=12(2^m)(2t+1)$ and $t=1, 2, \cdots$ has also been obtained, but will be discussed elsewhere. All the imbeddings are triangular, and hence an easy computation involving the Euler formula shows that the genus of K_n is $(n-3)(n-4)/12$ for $n=12s$, $s=1, 2, 3, \cdots$. For the connection between these results and the Heawood conjecture one may consult Ringel [3].

The method employed was announced by Gustin [2] and generalized by him to include the case in which the group used to name the vertices is non-Abelian, and "knobs" are permitted in the "quotient" network. The technique involves the delicate matching of a group to the geometry of a network, and in all other applications the group has been Abelian. On the other hand it can be shown that if the index of the solution is to be 1, as it is here, then the use of non-Abelian groups is essential.

The group. An appropriate group will be defined as the normal (or semi-direct) product (see [1, p. 88]) of a certain finite group and a group of its automorphisms. The finite group will be the *additive* group in a certain finite field and the *multiplicative* structure of the field will play a leading role in defining the automorphisms.

The following facts about finite fields will be used. (See [4, pp. 91–118].)

(1) For $k=1, 2, \cdots$ there is a finite field $GF(2^k)$, the Galois field of order 2^k.

(2) If $p\in GF(2^k)$ then $p+p=0$.

(3) The multiplicative group in $GF(2^k)$, here called $F^*(2^k)$, is cyclic. Suppose θ is a generator, then the order of θ is (2^k-1).

(4) If $F^+(2^k)$ is the additive group in $GF(2^k)$, then $1, \theta, \theta^2, \cdots, \theta^{k-1}$ is a basis for $F^+(2^k)$ over $F^+(2)$.

Using the exponential notation for an automorphism, define an automophism α of $F^+(2^k)$ by the linear extension of the following mapping of the generators:

$$(5)\qquad \begin{aligned} (\theta^i)^\alpha &= \theta^i, \qquad i = 2, \cdots, (k-1), \\ \theta^\alpha &= 1 + \theta, \\ 1^\alpha &= \theta. \end{aligned}$$

It follows that α is an automorphism of order 3. Hence the identity automorphism $e=\alpha^0$, α and α^2 form a group A. Form the normal product

$$F^+(2^k) * A.$$

This is a group $G(k)$, of order $3 \cdot 2^k$, with elements

$$[p, a]$$

where $p \in F^+(2^k)$ and $a \in A$. Group multiplication is defined by

$$[p, a] \cdot [q, b] = [p + q^a, ab].$$

The identity element is $[0, e]$, and $[p, a]^{-1} = [p^{a^{-1}}, a^{-1}]$.

The following properties of $G(k)$ are readily established by using (2) and (4).

(6) $[p, a]$ is of order 2 if and only if $p \neq 0$ and $a = e$.

(7) $[p, a]$ is of order 3 if and only if $p = c + d\theta$, $a \neq e$, and $c, d \in \mathrm{GF}(2)$.

Define

$$(8)\qquad \left.\begin{aligned} s_i &= [\theta^i + \theta^{i+1}, e], \\ t_i &= [\theta^i, \alpha], \end{aligned}\right\} \quad i = 1, \cdots, (2^k - 1), \qquad t_{2^k} = [0, \alpha].$$

The following statements are easily verified.

(9) The elements of order 2 are s_i, $i = 1, \cdots, (2^k - 1)$.

(10) t_1 and t_{2^k} are elements of order 3.

(11) Every element of $G(k)$ except the identity appears exactly once in the collection $s_i[i=1, \cdots, (2^k-1)]$, t_i, $t_i^{-1}[i=1, \cdots, 2^k]$.

$$(12)\qquad s_i \cdot t_i = t_{i+1}; \qquad i = 1, \cdots, (2^k - 2),$$

$$(13)\qquad t_i \cdot s_i = t_{i+1}; \qquad i = 2^k - 1.$$

It is not difficult to see that $G(k) \approx A_4 \times Z_2^m$ where $m = k - 2$, and A_4 is the alternating group on 4 objects. For small k, many other assignments for the s_i, and t_i have been obtained which satisfy statements (9) through (13). However, the only general method is the one above which uses the multiplicative structure of $\mathrm{GF}(2^k)$.

The quotient network. The quotient network for a triangular imbedding of K_n with $n=12\cdot 2^m$ is displayed below with currents from $G(k)$, $k=m+2$.

The nodes are displayed as dots, and each node, except those at the two ends, is of order 3. There are (2^k-1) "singular" arcs having but one node, and two "knobs" each having a node of order 1. (The singular arc to the right is the only one which lies above the horizontal line.)

The rotation at each node is counterclockwise. The circulation induced by the rotation has a single circuit, hence the index is 1.

As for the currents, the following properties hold and are *necessary and sufficient for a triangular imbedding.*

Each element of $G(k)$ except the identity, a total of $(n-1)$, appears exactly once on the circuit. (See (11).)

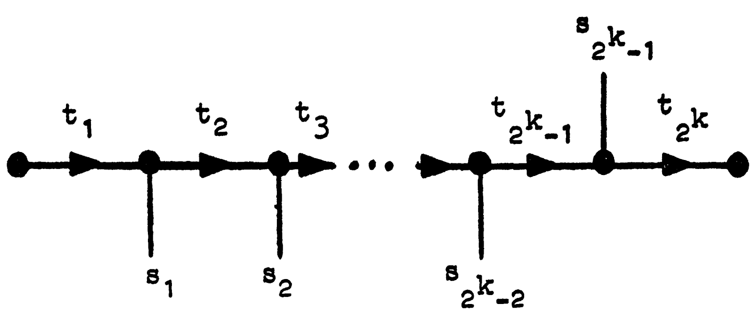
Incidentally, the order of their appearance provides a complete permutation of $G(k)-\{[0, e]\}$; namely,

$$(14)\quad (t_1 s_1 t_2\, s_2\, t_3 \cdots t_{2^k-2}\, s_{2^k-2}\, t_{2^k-1}\, t_{2^k}\, t_{2^k}^{-1}\, s_{2^k-1}\, t_{2^k-1}^{-1}\, t_{2^k-2}^{-1} \cdots t_3^{-1}\, t_2^{-1}\, t_1^{-1}).$$

The current on each singular arc is of order 2, and each element of order 2 is a current on some singular arc (see (9)).

The knobs carry currents of order 3 (see (10)).

At each node of order 3 the product of the *outward* directed currents taken in cyclic order of rotation is the identity (see (12) and (13)).



Concluding remarks. Define p_i to be the ith element in (14), $i=1, \cdots, (n-1)$, and $p_0=[0, e]$. Now consider the array

$$(15)\quad \begin{array}{lllll} p_0 & : p_1 & p_2 & p_3 & \cdots\; p_{n-1} \\ p_1 & : p_1^2 & p_1\cdot p_2 & p_1\cdot p_3 & \cdots\; p_1\cdot p_{n-1} \\ \vdots & & & & \\ p_{n-1} & : p_{n-1}\cdot p_1 & p_{n-1}\cdot p_2 & p_{n-1}\cdot p_3 & \cdots\; p_{n-1}^2 \end{array}$$

The technique of quotient networks *guarantees* that the array will be an orientable schema in the sense of Ringel (see [**3**, pp. 87–93] for an example with index 3). As Ringel has shown, if (15) is an orientable schema, then it provides a recipe for a minimal imbedding of K_n.

On the other hand, for (15) to be orientable it must satisfy his rule R^*: If, in the row identified by a, one has

$$a : \cdots b\ c\ d \cdots$$

then, in the row identified by c, one has

$$c : \cdots d\ a\ b \cdots .$$

A direct proof that a schema satisfies R^* is very laborious. (See the example quoted above.) Since the existence of a quotient network guarantees that R^* is satisfied, exposition can stop at that point! These comments indicate the power of the new technique.

Bibliography

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