GENERALIZED UNITARY OPERATORS[1]

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1. Let C be the complex field and Γ be the unit circle $\{\lambda \in C: |\lambda| = 1\}$. For a non-negative integer m or for $m = \infty$, let $C^m(\Gamma)$ be the space of all m-times continuously differentiable functions on Γ. (Here we consider Γ as a C^∞-manifold in the natural way. Thus, any $f \in C^m(\Gamma)$ can be identified with an m-times continuously differentiable periodic function $f(\theta)$ of a real variable θ with period 2π.) $C^m(\Gamma)$ is an algebra as well as a Banach space if m is finite, a Fréchet space if $m = \infty$, with the usual sup-norms for derivatives.

We shall say that a mapping γ of Γ into C is a *C^m-curve* if γ can be extended onto a neighborhood V of Γ (the extended map will also be denoted by γ) in such a way that it is one-to-one on V and γ and γ^{-1} are both m-times continuously differentiable (as functions in two variables) on V and $\gamma(V)$ respectively.

Let E be a Hausdorff locally convex space over C such that the space $\mathfrak{L}(E)$ of all continuous linear operators on E endowed with the bounded convergence topology is quasi-complete.

2. **$C^m(\gamma)$-operators.**

DEFINITION. Let γ be a C^m-curve. $T \in \mathfrak{L}(E)$ is called a *$C^m(\gamma)$-operator* if there exists a continuous algebra homomorphism W of $C^m(\Gamma)$ into $\mathfrak{L}(E)$ such that $W(1) = I$ and $W(\gamma) = T$. If γ is the identity map: $\gamma(\theta) = e^{i\theta}$, then a $C^m(\gamma)$-operator is called a *C^m-unitary operator*. (Cf. Kantrovitz' approach in [1].)

THEOREM 1. *If T is a $C^m(\gamma)$-operator, then* $\mathrm{Sp}(T) \subseteq \gamma(\Gamma)$.[2]

If H is a Hilbert space, $T \in \mathfrak{L}(H)$ is a C^0-unitary operator if and only if it is similar to a unitary operator on H. In this sense, C^m-unitary operators on E generalize the notion of unitary operators on a Hilbert space.

The homomorphism W in the above definition is uniquely determined by T and γ. Thus, we call W the $C^m(\gamma)$-representation for T. The uniqueness can be derived from the following approximation theorem: *Given a C^m-curve γ, let λ_0 be a point inside the Jordan curve*

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[2] $\mathrm{Sp}(T)$ is the spectrum of T in Waelbroeck's sense. See [2] for the definition.

$\gamma(\Gamma)$. *Then the set* $\{(P\circ\gamma)/(\gamma-\lambda_0)^n;\ P$: polynomial in one complex variable, n: integer $\geqq 0\}$ *is dense in* $C^m(\Gamma)$.

THEOREM 2. *Let* T *be a* $C^m(\gamma)$*-operator for a* C^m*-curve* γ *and let* W *be the* $C^m(\gamma)$*-representation for* T.

(i) *If* $A\in\mathfrak{L}(E)$ *commutes with* T, *then* A *commutes with each* $W(f)$, $f\in C^m(\Gamma)$.

(ii) *If* F *is a closed subspace of* E *left invariant under* T *and* $(\lambda_0 I-T)^{-1}$ *for some* λ_0 *inside of* $\gamma(\Gamma)$, *then it is left invariant under any* $W(f)$, $f\in C^m(\Gamma)$.

3. **Characterization theorem.** We recall ([2] and [4]) that $S\in\mathfrak{L}(E)$ with compact spectrum is called a C^m-scalar operator if there exists a continuous homomorphism U of the topological algebra[3] $C^m\equiv C^m(R^2)\equiv C^m(C)$ into $\mathfrak{L}(E)$ such that $U(1)=I$ and $U(\lambda)=S$.[4] In this case, the support of U is contained in $\mathrm{Sp}(S)$.

Now, we consider the following statements concerning $S\in\mathfrak{L}(E)$, depending on m and a C^m-curve γ:

$\mathrm{I}_\gamma(m)$: S is a $C^m(\gamma)$-operator.

$\mathrm{II}_\gamma(m)$: S is a C^m-scalar operator such that $\mathrm{Sp}(S)\subseteq\gamma(\Gamma)$.

$\mathrm{III}(m)$: $S^{-1}\in\mathfrak{L}(E)$ and for each continuous semi-norm q on $\mathfrak{L}(E)$, there exist a non-negative integer m_q ($=m$, if m is finite) and $M_q>0$ such that

$$(1)\qquad q(S^k)\leqq M_q|k|^{m_q}\qquad\text{for all } k=\pm 1,\pm 2,\cdots.$$

(Cf. [1].)

$\mathrm{IV}_\gamma(m)$: $\mathrm{Sp}(S)\subseteq\gamma(\Gamma)$ and for each continuous semi-norm q on $\mathfrak{L}(E)$, there exist a non-negative integer m_q ($=m$, if m is finite) and $M_q'>0$ such that

$$(2)\qquad q(R_\lambda)\leqq M_q' d_\lambda^{-m_q-1}\qquad\text{for all }\lambda\text{ with } 0<d_\lambda<1,$$

where $R_\lambda=(\lambda I-S)^{-1}$ for $\lambda\notin\mathrm{Sp}(S)$ and $d_\lambda=\mathrm{dis}(\lambda,\mathrm{Sp}(S))$. (Cf. [6].)

When γ is the identity map, we omit the subscript γ in the notations $\mathrm{I}_\gamma(m)$, $\mathrm{II}_\gamma(m)$ and $\mathrm{IV}_\gamma(m)$; in particular,

$\mathrm{I}(m)$: S is a C^m-unitary operator.

THEOREM 3 (THE CHARACTERISATION THEOREM).

(i) $\mathrm{I}(m)\Rightarrow\mathrm{II}(m)\Rightarrow\mathrm{III}(m)\Rightarrow\mathrm{IV}(m)\Rightarrow\mathrm{I}(m+2)$. *In particular,* $\mathrm{I}(\infty)$, $\mathrm{II}(\infty)$, $\mathrm{III}(\infty)$ *and* $\mathrm{IV}(\infty)$ *are mutually equivalent.*

(ii) $\mathrm{I}_\gamma(m)\Rightarrow\mathrm{II}_\gamma(m)\Rightarrow\mathrm{IV}_\gamma(m)\Rightarrow\mathrm{I}_\gamma(m+2)$.[5]

[3] C^m is the space of all m-times continuously differential functions on $R^2=C$. The topology in it is defined by sup. of derivatives on compact sets.

[4] λ denotes the identity function $f(\lambda)\equiv\lambda$.

[5] In the implication $\mathrm{IV}_\gamma(m)\Rightarrow\mathrm{I}_\gamma(m+2)$, we are assuming that γ is a C^{m+2}-curve.

In particular, $\mathrm{I}_\gamma(\infty)$, $\mathrm{II}_\gamma(\infty)$ *and* $\mathrm{IV}_\gamma(\infty)$ *are mutually equivalent.*

4. Here, we shall give indications of proofs of Theorem 3, (i). The proofs of (ii) are similar but more complicated.

$\mathrm{I}(m)\Rightarrow\mathrm{II}(m)$: If W is the $C^m(e^{i\theta})$-representation for S, then we define $U(\phi)=W(\phi(e^{i\theta}))$ for $\phi\in C^m$. Then U is a C^m-representation for S.

$\mathrm{II}(m)\Rightarrow\mathrm{III}(m)$: Since $S^k=U(\lambda^k)$, we obtain (1) evaluating the norms of λ^k on neighborhoods of Γ and using the continuity of U.

$\mathrm{III}(m)\Rightarrow\mathrm{IV}(m)$: If $|\lambda|<1$, then $R_\lambda=-\sum_{k=0}^\infty \lambda^k S^{-(k+1)}$; if $|\lambda|>1$, then $R_\lambda=\sum_{k=0}^\infty \lambda^{-(k+1)}S^k$. Hence, (2) follows from (1).

$\mathrm{IV}(m)\Rightarrow\mathrm{I}(m+2)$: For $f\in C^{m+2}(\Gamma)$, we define

$$W(f)=\lim_{\epsilon\to 0+}\frac{1}{2\pi}\left\{\int_0^{2\pi} f(\theta)\left[R_{(1+\epsilon)e^{i\theta}}-R_{(1-\epsilon)e^{i\theta}}\right]e^{i\theta}d\theta\right\}.$$

By a method due to Tillmann ([5] and [6]), we see that the right-hand side is well-defined and that W is the $C^{m+2}(e^{i\theta})$-representation for S.

5. Corollary and examples.

COROLLARY. *If* S_i $(i=1,2)$ *is a* C^{m_i}*-unitary operator and if* S_1 *and* S_2 *commute, then* S_1S_2 *is a* $C^{m_1+m_2+2}$*-unitary operator.*

This is a consequence of Theorem 2, Theorem 3 and the corollary to Proposition 3.1 of [3].

EXAMPLES. Let $\mathcal{S}(R^n)$ be the Fréchet space of rapidly decreasing functions on R^n. $[\mathcal{S}(R^n)]'$ is the space of tempered distributions. Let $E=\mathcal{S}(R^n)$ or $[\mathcal{S}(R^n)]'$. The translations τ_α: $[\tau_\alpha f](x)=f(x+\alpha)$ are C^∞-unitary operators on E; the Fourier transform is a C^2-unitary operator on E.

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