THE EXTREMAL FUNCTIONS FOR CERTAIN PROBLEMS CONCERNING SCHLICHT FUNCTIONS

BY ARTHUR E. OBROCK[1]

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1. Let S denote the classical family of schlicht functions on the unit disk normalized by the conditions $f(0)=0$, $f'(0)=1$. Under a suitable metric such as $d(f, g)=\sup\{f(z)-g(z)| : |z|=1/2\}$ it is a compact metric space. Let $0<r<1$. We are interested in the closed subspaces $B_r=\{f\in S: |f(z)|<1/r\}$ and $C_r=\{f\in S: f(z)\notin D(f)\}$, where $D(f)$ is a domain of outer conformal radius $1/r$ with respect to the point at infinity. The general problem is to determine the explicit region of values $V(T)$ of certain continuous functions T from one of these spaces F into some manifold M. We also ask what (extremal) functions in F are mapped by T into $\partial V(T)$, the boundary of $V(T)$. In particular consider the function

$$(*) \quad T(f) = (f^0(z_1), f^1(z_1), \cdots, f^{n_1}(z_1), \cdots, f^0(z_m), \cdots, f^{n_m}(z_m)),$$

where $f^k(z_j)=H(f, z_j, k)$ denotes the value of the kth derivative of f at z_j, except that, for technical reasons H is interpreted as a continuous function into the logarithmic covering surface when $z_j\neq 0$ and $k=0$ or 1.

The well-known results for the case $F=S$, $m=1$, $z_1=0$, due to Spencer and Schaeffer, can be found in [10]. Royden [11] indicated the more general result when $F=S$. Their key tools were Teichmüller's Theorem [10, p. 93] and their variational method. By using Jenkins' General Coefficient Theorem [7] and a form of the Brouwer Fixed Point Theorem we are able to generalize some of their results to a somewhat wider class of functions T and spaces F.

2. For the functions T defined by (*) there are certain quadratic differentials $P(w)dw^2$, indicated by the Teichmüller Principle [8, p. 48], which we call *admissible with respect to* T. We call the pair $(P(w)dw^2, f(z))$ an *admissible association with respect to* T if $P(w)dw^2$ is admissible with respect to T, $f\in F$, $f(\{|z|<1\})$ is an admissible domain with respect to $P(w)dw^2$ in the sense of Jenkins [8, p. 49], and $\{f(z_j): 1\leq j\leq m\}$ contains the poles of $P(w)dw^2$ in $f(\{|z|<1\})$.

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Now we can state our main theorem.

THEOREM. *Let T be defined by (*). To every boundary point p_* of $V(T)$ there is precisely one function f_* in F such that $T(f_*)=p_*$. There is at least one admissible association with respect to $T(P(w)dw^2, f_*(z))$. Conversely, if $(P(w)dw^2, f(z))$ is an admissible association with respect to T then $T(f)$ lies in the boundary of $V(T)$.*

OUTLINE OF THE PROOF. It is easy to show that $V(T)$ is homeomorphic to a unit ball in a Euclidean space of an appropriate dimension and that an infinite number of functions in F are mapped by T into every interior point of $V(T)$ by a proof analogous to the proof in [**10**, p. 9] for the (Bieberbach) coefficient region. Consequently, the last statement of our theorem follows from the equality statement of the General Coefficient Theorem. Next we introduce a topology on W, the set of admissible associations with respect to T. Then we construct a map $H_1: \partial V(T)\to W$ such that the composed map $H_2 \circ H_1$: $\partial V(T)\to\partial V(T)$, (where H_2: $(P(w)dw^2, f(z))\to T(f)$) is homotopic to the identity relative to $\partial V(T)$. The remainder of the theorem then follows from a form of the Brouwer Fixed Point Theorem.

3. The computation of the explicit values of T is in general a difficult task. One approach is to find the relations between the coefficients of $P(w)dw^2$ and the coefficients of $Q(z)dz^2=P(f(z))(f'(z))^2dz^2$, the induced quadratic differential. Then the values of $T(f)$ can be computed in terms of these coefficients. In simple cases we are able to obtain the explicit results in this way. For example we obtain a generalization of Grunsky's region $V(T)$ [**5**] (where $T(f)=\log f(\rho e^{i\theta})$, $F=S$) to the spaces B_r, C_r.

In the case B_r $V(T)$ is the convex region bounded by the Jordan curve $C(t)=\log(s(t)/r)+i(\theta+\phi(t))$, for $-\pi\leqq t\leqq\pi$, where $s=s(t)=s(-t)$, $\phi=\phi(t)=-\phi(-t)$, and are defined on $0\leqq t\leqq\pi$ from the homeomorphisms indicated by the relations

$$\cos t = \log\frac{1-\rho^2}{1-s^2}\,\frac{s}{r\rho}\Big/\log\frac{1-\rho}{1+\rho}\,\frac{1+s}{1-s},\quad \phi = \sin t\log\frac{1+\rho}{1-\rho}\,\frac{1-s}{1+s}\,.$$

The unique extremal function $f_t\in B_r$ associated with $C(t)$ is the unique function such that $f_t(\{|z|<1\})$ is admissible with respect to

$$P(w)dw^2 = -e^{-it}\frac{(w-e^{i(\phi+t)}/r)^2dw^2}{w^2(w-se^{i\phi}/r)(w-e^{i\phi}/sr)}\,.$$

When such explicit solutions are determined, we can often follow Jenkins' example [**9**] to solve problems where $T_1=H\circ T$. For example, if $T_1(f)=\arg(f(\rho e^{i\theta})/\rho e^{i\theta})$ on B_r, then $V(T_1)$ is found by determining $\sup\{\phi(t)\,|\,0\leqq t\leqq\pi\}$. We find that if $f\in B_r$, then

$$|\arg(f(\rho e^{i\theta})/\rho e^{i\theta})| \leqq \left(\log\frac{1-s_*}{1+s_*}\,\frac{1+\rho}{1-\rho}\right)\left(\frac{1-s_*^2}{1+s_*^2}\right),$$

where $s_*=s(t_*)$ is the unique value of s where

$$2\log\frac{1-\rho}{1+\rho}\,\frac{1+s}{1-s}=\left(s+\frac{1}{s}\right)\log\frac{1-\rho^2}{1-s^2}\,\frac{s}{r\rho}\,.$$

Equality occurs only for the functions f_{t^*}, f_{-t^*}.

4. Duren and Schiffer [2], Gaier and Huckemann [3], Huckemann [6], Gehring and Hällström [4], and Duren [5] have investigated schlicht functions on an annulus. More precisely, let F_r denote the family of schlicht functions on $r<|z|<1/r$ such that $f(1/\bar{z})=1/\bar{f}(z)$, $f(1)=1$. We obtain results similar to those of the theorem in §2 for functions T on F_r of the form (*). In the quadratic differentials $P(w)dw^2$, $P(w)$ is again a rational function of w. However, the computations of $T(f)$ are more complicated here due to the fact that in the induced $Q(z)dz^2$, $Q(z)$ is no longer rational in z but is given by $v(\log z)/z^2$ where $v(w)$ is a doubly periodic function.

Bibliography

1. P. L. Duren, *Distortion in certain conformal mappings of an annulus*, Michigan Math. J. **10** (1963), 431–441.

2. P. L. Duren and M. Schiffer, *A variational method for functions schlicht in an annulus*, Arch. Rational Mech. Anal. **9** (1962), 260–272.

3. D. Gaier and F. Huckemann, *Extremal problems for functions schlicht in an annulus*, Arch. Rational Mech. Anal. **9** (1962), 415–421.

4. F. W. Gehring and G. af Hällström, *A distortion theorem for functions schlicht in an annulus*, Ann. Acad. Sci. Fenn. Ser. A I (1963), no. 325, 15 pp.

5. H. Grunsky, *Neue Abschätzungen zur konformen Abbildung ein- und mehrfach zusammenhängender Bereiche*, Schr. Deutsche Math.-Ver. **43** (1934), 140–143.

6. F. Huckemann, *Über einige Extremalprobleme bei konformer Abbildung eines Kreisringes*, Math. Z. **80** (1962), 200–208.

7. J. A. Jenkins, *A general coefficient theorem*, Trans. Amer. Math. Soc. **77** (1954), 262–280.

8. ———, *Univalent functions and conformal mapping*, Springer-Verlag, Berlin, 1958.

9. ———, *On weighted distortion in conformal mapping*, Illinois J. Math. **4** (1960), 28–37.

10. A. C. Schaeffer and D. C. Spencer, *Coefficient regions for schlicht functions*, Amer. Math. Soc. Colloq. Publ. Vol. 35, Amer. Math. Soc., Providence, R. I., 1950.

11. H. L. Royden, *The interpolation problem for schlicht functions*, Ann. of Math. **60** (1954), 326–344.

University of Illinois